Sur la théorie des représentations et les algèbres d'opérateurs des produits en couronnes libres

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Sur la théorie des représentations et les algèbres d'opérateurs des produits en couronnes libres

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“J’ai beaucoup travaillé.
Quiconque travaillera comme moi pourra faire ce que j’ai fait.”

Jean-Sébastien Bach

“Je ne connais pas la proportion exacte mais j’ai toujours pensé que pour chaque heure passée en compagnie d’êtres humains, il fallait x heures pour travailler seul. Ce qu’est x, je l’ignore, deux heures et sept huitièmes ou sept heures et deux huitièmes mais c’est une quantité considérable.”

Glenn Gould

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SUR LA THÉORIE DES REPRÉSENTATION ET LES ALGÈBRES D'OPÉRATEURS DES PRODUITS EN COURONNES LIBRES

Résumé

par FRANÇOIS LEMEUX

Dans cette thèse, on étudie les propriétés combinatoires, algébriques et analytiques de certains groupes quantiques compacts libres. On prouve au Chapitre 2 que les duaux des groupes quantiques de réflexions complexes H_N^{s+} possèdent la propriété d'approximation de Haagerup pour tout $N \geq 4$ et $s \in [1, +\infty)$. On utilise la surjection canonique $\pi : C(H_N^{s+}) \to C(S_N^+)$ et on décrit comment les caractères de $C(H_N^{s+})$ se comportent via ce morphisme π grâce à la description des règles de fusion liant les coreprésentations irréductibles de H_N^{s+} calculée par Banica et Vergnioux. Cela nous permet de construire des états sur la C^*-algèbre centrale $C(H_N^{s+})_0$ engendrée par les caractères de $C(H_N^{s+})$ et d'utiliser un théorème fondamental de Brannan proposant une méthode de construction d'applications normales, unitales, complètement positives et préservant la trace sur l'algèbre de von Neumann d'un groupe quantique compact de type Kac.

Au Chapitre 3, on décrit les règles de fusion des produits en couronnes libres $\widehat{\Gamma} \wr_* S_N^+$ pour tout groupe discret Γ. Pour cela on détermine les espaces d'entrelaceurs entre certaines coreprésentations "basiques" de ces produits en couronnes libres en termes de partitions non-croisées décorées par les éléments du groupe Γ. On peut alors identifier les coreprésentations irréductibles et décrire les règles de fusion des produits en couronnes libres $\widehat{\Gamma} \wr_* S_N^+$. On propose ensuite plusieurs applications de ce résultat. On démontre premièrement que la C^*-algèbre réduite des produits en couronnes libres $\widehat{\Gamma} \wr_* S_N^+$ est dans la plupart des cas simple et à trace unique. On adapte pour cela une méthode de Powers utilisée par Banica pour montrer la simplicité de $C_r(U_N^+)$ et on utilise la simplicité de $C_r(S_N^+)$ pour tout $N \geq 8$, établie par Brannan [Bra12b]. Puis on prouve que l'algèbre de von Neumann associée est un facteur de type II_1 et que ce facteur est plein. Pour cela, on adapte la méthode classique des "$14 - \epsilon$" utilisée pour démontrer que les facteurs du groupe libre $L(F_N)$ ne possèdent pas la propriété Γ. On montre finalement que les produits en couronnes libres $\widehat{\Gamma} \wr_* S_N^+$ possèdent la propriété de Haagerup pour tout groupe Γ fini.

ON THE THEORY OF REPRESENTATIONS AND THE OPERATOR ALGEBRAS OF FREE WREATH PRODUCTS

Abstract

by FRANÇOIS LEMEUX

In this thesis, we study the combinatorial and operator algebraic properties of certain free compact quantum groups. We prove in Chapter 2 that the duals of the quantum reflection groups H_N^{s+} have the Haagerup property for all $N \geq 4$ and $s \in [1, \infty)$. We use the canonical arrow $\pi : C(H_N^{s+}) \to C(S_N^+)$ onto the quantum permutation group, and we describe how the characters of $C(H_N^{s+})$ behave with respect to this morphism π thanks to the description by Banica and Vergnioux of the fusion rules binding irreducible corepresentations of H_N^{s+}. This allows us to construct states on the central C^*-algebra $C(H_N^{s+})_0$ generated by the characters of $C(H_N^{s+})$ and to use a fundamental theorem proved by Brannan giving a method to construct nets of trace-preserving, normal, unital and completely positive maps on the von Neumann algebra of a compact quantum group of Kac type.

In Chapter 3, we describe the fusion rules of the free wreath products $\widehat{\Gamma} \wr_* S_N^+$ for all discrete groups Γ. To do this we describe the intertwinner spaces between certain "basic" corepresentations of these free wreath products in terms of non-crossing partitions decorated by the elements of the group Γ. This provides a whole new class of compact quantum groups whose fusions rules are explicitly computed. We give several applications to this result. We prove that in most cases the reduced C^*-algebra associated with these free wreath products is simple with unique trace. To do this, we adapt Powers' methods used by Banica and we use the simplicity of $C_r(S_N^+)$ for all $N \geq 8$ proved by Brannan. We also prove that the associated II_1-factor $L^\infty(\widehat{\Gamma} \wr_* S_N^+)$ is full. This is adapted from the "$14 - \epsilon$" method used to prove that the free group factor $L(F_N)$ doest not have the property Γ. To conclude, we extend the result of Chapter 2, proving that the duals of the free wreath products $\widehat{\Gamma} \wr_* S_N^+$ have the Haagerup property for all *finite* groups Γ.

Contents

Introduction

Motivations Dans cette première partie, je donne quelques motivations de mon travail et tente de replacer mes recherches dans le contexte mathématique dans lequel je les développe. Mes recherches se situent dans le domaine des algèbres d'opérateurs c'est à dire des algèbres d'applications linéaires continues sur des espaces vectoriels topologiques. Lorsque l'espace vectoriel en jeu est un espace de Hilbert, l'application adjointe sur les opérateurs produit alors une involution naturelle sur ces algèbres. Les algèbres d'opérateurs auto-adjointes sont un sujet d'étude depuis les années 1930 avec notamment les travaux de Murray et von Neumann. Ces objets recouvrent à la fois les C^*-algèbres et les algèbres de von Neumann. Des caractérisations simples en termes de normes permettent de caractériser les C^*-algèbres au sein des algèbres d'opérateurs auto-adjointes. Il est également remarquable que les algèbres de von Neumann admettent une caractérisation algébrique simple en terme de bi-commuttant. On pourra se référer à [Jon10], [Con00], [Con90b] et [BO08] pour des introductions aux algèbres d'opérateurs et aux outils nécessaires à leur étude.

Les algèbres de von Neumann peuvent être caractérisées comme des sous-algèbres fermées pour la topologie ultrafaible d'espace d'opérateurs $B(H)$, avec H un espace de Hilbert. Comme rappelé ci-dessus, elles sont également caractérisées algébriquement. Ces deux aspects, analytique et algébrique, permettent d'aborder les questions concernant ces objets sous différents angles. C'est le cas pour les problèmes liés à leur classification. La classification des algèbres de von Neumann se ramène à celle des facteurs, c'est à dire les algèbres dont le centre est trivial. On distingue alors trois types de facteurs selon les propriétés des projections qu'ils contiennent : types I, II, III. Les facteurs de type I sont isomorphes à un $B(H)$ pour un certain espace de Hilbert H et les algèbres de matrices en sont donc des exemples, très élémentaires. Le problème beaucoup plus difficile de la classification des facteurs de type III a été résolu par Connes dans les années 1970 [Con73]. Les facteurs de II se décomposent en deux sous-familles, ceux de type II_1 et de type II_∞, ces derniers étant en fait des produits tensoriels de facteurs de type I et de type II_1. Un sujet particulièrement intéressant est donc la classification des facteurs de type II_1. Ce problème est en fait très difficile et a été source de nombreux développements. Les premiers résultats ont été obtenus par Murray et von Neumann qui ont montré en particulier qu'il existe un unique facteur hyperfini de type II_1, c'est à dire réunion croissante de sous-C^*-algèbres de dimensions finies. Un tel facteur hyperfini peut être obtenu en considérant l'algèbre de von Neumann $L(G)$ d'un groupe moyennable infini à classes de conjugaisons infinies, $G = S_\infty$ par exemple. En fait, tout groupe discret ayant la propriété des classes de conjugaisons infinies tel que le groupe libre non-abélien

$\mathbb{F}_n$, $n \geq 2$, produit un exemple de facteur de type II_1. Mais Murray et von Neumann ont montré que pour aucun $n \geq 2$, $L(\mathbb{F}_n)$ n'est isomorphe à $L(S_\infty)$. Au delà de ces exemples, la difficulté de classifier les facteurs de type II_1 s'illustre par exemple dans le problème toujours ouvert de l'isomorphisme des facteurs du groupe libre : est-il vrai que $L(\mathbb{F}_n) \simeq L(\mathbb{F}_m)$ pour différentes valeurs de n et m ?

Une avancée significative sur ce problème d'isomorphisme est liée à la notion de groupe fondamental qui produit un invariant pour les facteurs de type II_1 ; il s'agit d'un sous-groupe de $\mathbb{R}^+$. Précisément le groupe fondamental d'un facteur de type II_1 muni d'une trace τ, (M, τ), est l'ensemble des réels $t > 0$ tels que $M_t \simeq M$ où $M_t = pMp$ avec $p \in M$ une projection de trace t. Dykema [Dyk94] et Rădulescu [Răd94] ont montré, indépendamment, que tous les facteurs du groupe libre interpolé $L(\mathbb{F}_r)$ sont soit tous mutuellement isomorphes soit non-isomorphes deux-à-deux. Ce résultat repose sur l'étude de leur groupe fondamental. Malheureusement, on ne sait pas décider si le groupe fondamental des facteurs du groupes libres (interpolés) sont tous égaux à $\mathbb{R}^+$ ou bien tous triviaux (et le précédent résultat repose en fait sur cette alternative). De manière général, on connait peu les groupes fondamentaux des facteurs de type II_1. Principalement, on sait que $\mathscr{F}(L(S_\infty)) = \mathbb{R}^+$ et qu'on a également $\mathscr{F}(L(\mathbb{F}_\infty)) = \mathbb{R}^+$. Popa a donné un exemple où le groupe fondamental est trivial [Pop04].

Ces avancées trouvent leur cadre dans le domaine des probabilités libres. Les facteurs du groupe libre interpolés $L(\mathbb{F}_r)$ proviennent d'une famille continue de facteurs de type II_1, qui est en correspondance aux valeurs entières avec les facteurs du groupe libre usuels et est construite à partir d'une famille infinie d'éléments semi-circulaires libres d'un espace de probabilité non-commutatif. La théorie des probabilités non-commutatives a été introduite par Voiculescu. Elle repose sur la notion d'$*$-algèbre et de variables aléatoires non-commutatives libres, l'analogue non-commutatif de l'indépendance des variables aléatoires classiques, voir par exemple [Voi95]. Cette théorie s'est développée depuis les années 1980 et a donné des résultats remarquables comme ceux rappelés ci-dessus. Les concepts de matrices aléatoires et de liberté asymptotique sont centraux dans ces questions et le calcul des moments de variables aléatoires est donc essentiel pour les appréhender. Speicher et Nica ont, en particulier, donné des formules permettant le calcul de tels moments via la notion de cummulants libres [NS06]. Les méthodes utilisées sont très combinatoires et les objets centraux sont les partitions non-croisées.

Les groupes quantiques compacts libres sont à la croisée des différentes problématiques évoquées ci-dessus. En effet, on peut étudier ces groupes quantiques du point de vue des algèbres d'opérateurs, c'est l'un des objets des prochains chapitres de cette thèse ; mais on peut également étudier ces groupes quantiques d'un point de vue combinatoire et ce sera aussi l'objet d'une partie du chapitre 3. En effet, Banica puis Banica et

Collins, Banica et Speicher ont initié l'étude de certaines familles de groupes quantiques compacts de ce point de vue [Ban97], [BC07], [BS09] : on sait par exemple que l'état de Haar sur O_N^+ est par déterminé par les coefficients d'une matrice de Weingarten. Les groupes quantiques étudiés dans de cette thèse ont de plus la propriété remarquable que les espaces d'entrelaceurs entre leurs coreprésentations sont encodés par certaines partitions non-croisées, voir Chapitre 1 Sous-section 1.1.6 et Chapitre 3 Theorem 3.2.12, Theorem 3.2.20.

Le dualité de Tannaka-Krein obtenue par Woronowicz dans [Wor88] montre que les groupes quantiques sont fortement déterminés par leurs espaces d'entrelaceurs. La description combinatoire de ces entrelaceurs est donc un outil puissant, et en fait essentiel, pour approfondir la connaissance des groupes quantiques et par conséquent des algèbres d'opérateurs associées, voir ci-dessous.

Notons avant de poursuivre que le lien entre les groupes quantiques et les probabilités libres est illustré également au travers d'un analogue non-commutatif du théorème de De Finetti [KS09]. Ce dernier stipule que dans un espace de probabilité non-commutatif (A, ϕ), la distribution jointe d'une suite infinie d'élément $(x_k)_{k\in\mathbb{N}}$ est invariante par permutations quantiques si et seulement si la suite est i.i.d et libre par rapport à l'espérance conditionnelle sur $\bigcap_{n\geq 1} vN(x_k : k \geq n)$, au sens de Voiculescu. Les récents résultats de classifications sur les groupes quantiques easy (voir par exemple [Web13], [RW12], [FW13]) ouvrent alors le champ à l'étude de symétries quantiques associées à d'éventuelles nouvelles distributions non-commutatives.

Comme évoquée plus haut, la connaissance des espaces d'entrelaceurs de certains groupes quantiques compacts a permis de déterminer leurs coreprésentations irréductibles et le calcul de leurs règles de fusion ; c'est à dire de donner des formules explicites permettant le calcul de produits tensoriels entre coreprésentations irréductibles, voir [Ban96], [Ban97], [Ban99b],[BV09]. Les propriétés combinatoires de ces règles de fusion sont en quelque sorte le pendant non-commutatif des propriétés combinatoires des groupes discrets classiques. En suivant cette idée, la connaissance des règles de fusion de certains groupes quantiques compacts a permis d'étudier les propriétés des algèbres d'opérateurs associées telles que la (non-) co-moyennabilité [Ban97], la propriété de décroissance rapide [Ver07] ainsi que les propriétés de factorialité, d'Akemann-Ostrand et d'exactitude [Ver05], [VV07] pour les groupes quantique libres orthogonaux et unitaires dans la plupart des cas. En outre, en vue des résultats prouvés au Chapitre 3 de cette thèse, il est intéressant de noter que les méthodes de Banica pour prouver la simplicité de $C_r(U_N^+)$ sont inspirées de celles de Powers pour démontrer la simplicité de $C_r(\mathbb{F}_N)$. On retrouve à nouveau l'idée que l'étude des propriétés combinatoires des groupes classiques se traduit dans le cas quantique au niveau des propriétés combinatoires des règles de fusion.

Plus récemment, Brannan a obtenu la propriété de Haagerup pour les algèbres de von Neumann associées aux groupes quantiques compact libres orthogonaux et unitaires ainsi que pour les groupes quantiques de permutations [Bra12a], [Bra12b]. On détaille certains de ces résultats dans le Chapitre 1. On réalise donc que les propriétés d'algèbres d'opérateurs partagées par les facteurs des groupes libres et les algèbres de von Neumann associées à ces groupes quantiques libres sont nombreuses. De récents résultats pour les groupes quantiques mettent à nouveau ces objets en parallèle. On sait depuis longtemps que $L(\mathbb{F}_N)$ n'a pas de sous-algèbre de Cartan, [Voi96]. Ce résultat a été généralisé par Popa et Ozawa grâce à des techniques de déformation/rigidité qui ont produit des résultats de structures très profonds. En particulier, les facteurs du groupe libre sont fortement solides. Cela redémontre qu'ils n'admettent pas de sous-algèbres de Cartan et également qu'ils ne peuvent pas s'écrire comme produit tensoriel de facteurs de dimension infinie. Ces techniques sont aussi à l'origine de résultats sur les sous-algèbres de Cartan de certains produits croisés provenant d'actions de groupes hyperboliques [VP11]. Suivant ces idées, Isono a donné une condition suffisante de solidité forte pour des algèbres de von Neumann ne provenant pas nécessairement des groupes. En combinant ces résultats avec la moyennabilité faible prouvée par Freslon pour les groupes quantiques libres orthogonaux, il a montré l'absence de sous-algèbres de Cartan pour $L^\infty(O_N^+)$. Depuis, on connait d'autres résultats concernant la moyennabilité faible des algèbres d'opérateurs associées à certains groupes quantiques compact libres [DCFY13]. Des résultats structurels tels que la bi-exactitude étant déjà connus, les auteurs de [DCFY13] obtiennent dans certains cas (des hypothèses techniques assurant la non-injectivité), l'absence de sous-algèbres de Cartan pour les algèbres de von Neumann associées à des groupes quantiques compacts libres non nécessairement de type Kac. On détaille ceci au Chapitre 1 et on fait le lien avec les résultats présentés dans les autres chapitres.

Dans le prochain paragraphe, je détaille le plan de cette thèse et j'explique comment les résultats de celle-ci s'intègrent dans la lignée des travaux décrits ci-dessus.

Plan de la thèse L'un des objectif de mes travaux est de poursuivre l'étude des algèbres d'opérateurs associées aux groupes quantiques compacts libres. On montre dans le Chapitre 2 que les algèbres de von Neumann des groupes quantiques de réflexions complexes H_N^{s+}, ont la propriété de Haagerup pour tout $N \geq 4$ et tout $s \in [1, +\infty)$, Théorème 2.3.5. Ce résultat donne un nouvel exemple d'une famille de groupes quantiques compacts vérifiant cette propriété. Jusqu'alors, cette propriété n'était connue que pour les algèbres de von Neumann des groupes quantiques orthogonaux et unitaires libres et des groupes d'automorphismes quantiques des C^*-algèbres de dimension finie munies d'une trace. Ce résultat est également motivé par un résultat provenant des groupes discrets classiques : la propriété de Haagerup est stable pour le produit en couronne de groupes discrets classiques [CSV12]. En effet, les groupes quantique de réflexions sont des analogues quantiques de groupes de réflexions complexes H_N^s : d'une part la C^*-algèbre $C(H_N^s)$ est un quotient de $C(H_N^{s+})$ et d'autre part, on a la formule suggestive $H_N^{s+} \simeq \mathbb{Z}/s\mathbb{Z} \wr_* S_N^+$, analogue du produit en couronne classique $H_N^s = \mathbb{Z}/s\mathbb{Z} \wr S_N$. La question de la stabilité dans le cas des groupes quantiques est donc naturelle et l'on y apporte une réponse partielle dans cette thèse, au Chapitre 2.

Pour démontrer la propriété de Haagerup, on utilise un résultat fondamental de Brannan [Bra12a] qui propose une méthode de construction d'applications complètement positives sur l'algèbre de von Neumann d'un groupe quantique de type Kac à partir d'états sur l'algèbre centrale. Plus précisément, partant d'un groupe quantique $\mathbb{G}$ de type Kac, et d'un état ψ sur l'algèbre engendrée par les caractères χ_α des coreprésentations irréductibles, le théorème de Brannan montre que l'application

$$T_\psi = \sum_{\alpha \in Irr(\mathbb{G})} \frac{\psi(\chi_{\bar{\alpha}})}{d_\alpha} p_\alpha$$

est une application normale, unitaire, complètement positive sur $L^\infty(\mathbb{G})$. Dans le cas où $\mathbb{G} = H_N^{s+}$, l'objectif est donc de déterminer une suite d'états appropriée sur la C^*-algèbre $C(H_N^{s+})_0$ engendrée par les caractères irréductibles de H_N^{s+}. Pour cela on utilise la surjection canonique $\pi : C(H_N^{s+}) \to C(S_N^+)$ à laquelle correspond un foncteur π : $Rep(H_N^{s+}) \to Rep(S_N+)$ associant à la matrice génératrice U de $H_N^+(\Gamma)$, la génératrice v de S_N^+. On détermine alors via cette flèche l'image des caractères irréductibles de H_N^{s+}, voir Proposition 2.2.1. Avec les notations du Chapitre 2, le caractère χ_α, $\alpha = a^{l_1} z_{j_1} \dots a^{l_k}$, s'envoie sur le produit des polynômes de Tchebytchev $\prod_i A_{l_i}(\sqrt{x})$.

On obtient alors pour tout $x \in (0, N)$ un état $\psi_x \in C(H_N^{s+})_0^*$ par composition $ev_x \circ \pi$, où $ev_x \in C(S_N^+)_0^* \simeq C([0, N])^*$. En faisant $x \to N$, on voit alors que ces états convergent vers la counité et l'on en déduit facilement que les applications T_{ψ_x} convergent vers l'identité point par point en norme 2. Pour montrer que les extensions L^2, encore notées T_{ψ_x}, sont compactes, il suffit de montrer que les valeurs propres $\dfrac{\psi(\chi_{\bar{\alpha}})}{d_\alpha}$ de cet opérateur diagonal par bloc sont dans $c_0(Irr(\mathbb{G}))$ (en effet, les projections p_α sont de rang fini). Pour montrer cela, on utilise les estimés de la Proposition 2.1.7 : pour tout $N \geq 2$, et tout $x \in (2, N)$, il existe une constante $c \in (0, 1)$ telle que pour tout entier $t \geq 1$,

$$0 < \frac{A_t(x)}{A_t(N)} \leq \left(\frac{x}{N}\right)^{ct}.$$

On en déduit alors que pour $N \geq 5$ et tout $x \geq 5$,

$$\frac{\psi_x(\chi_{\bar{\alpha}})}{d_\alpha} = \prod_i \frac{A_{l_i}(\sqrt{x})}{A_{l_i}(\sqrt{N})} \leq \left(\frac{x}{N}\right)^{\frac{c}{2}\sum l_i}.$$

Cette quantité converge vers 0 lorsque $\sum_{l_i} \to \infty$. Le cardinal du groupe $\mathbb{Z}/s\mathbb{Z}$ étant fini, on en déduit que la fonction $\sum_i l_i$ est propre sur $Irr(\mathbb{G})$ et par suite que

$$\frac{\psi_x(\chi_{\bar{\alpha}})}{d_\alpha} = \prod_i \frac{A_{l_i}(\sqrt{x})}{A_{l_i}(\sqrt{N})} \in c_0(Irr(\mathbb{G})).$$

Les cas particuliers $N = 2$ et $N = 4$ sont également traités dans le Chapitre 2. Le cas $N = 3$ reste ouvert. En effet, les règles de fusion ne sont pas connues dans ce cas, et on ne parvient pas à exprimer $C(H_3^{s+})$ plus simplement comme dans le cas $N = 2$.

Dans le Chapitre 3, je détermine dans la section 3.2 les règles de fusion des produits en couronnes libres $\widehat{\Gamma} \wr_* S_N^+$ pour tout groupe discret classique Γ. Pour cela, on commence par déterminer les espaces d'entrelaceurs dans le produit libre de groupes quantiques $(H_N^{\infty+})^{*p}$ où p désigne le cardinal d'une partie génératrice du groupe Γ. On en déduit alors, en étudiant le noyau du morphisme $\rho : C(H_N^{\infty+})^{*p} \to C(H_N^+(\Gamma))$, que les espaces d'entrelaceurs dans $H_N^+(\Gamma)$ entre les coreprésentations dites basiques $a(g) = (g^{(i)}v_{ij})$, $g \in \Gamma$, sont donnés (au Théorème 3.2.20) par

$$\begin{aligned} Hom_{H_N^+(\Gamma)}&(a(g_1) \otimes \cdots \otimes a(g_k); a(h_1) \otimes \cdots \otimes a(h_k)) \\ &= span\{T_p : p \in NC_\Gamma(g_1, \ldots, g_k; h_1, \ldots, h_l)\}. \end{aligned}$$

On a noté $NC_\Gamma(g_1, \ldots, g_k; h_1, \ldots, h_l)$ l'ensemble des partitions non-croisées dans $NC(k, l)$ telles que dans chaque bloc, $\prod_i g_i = \prod_j h_j$ et T_p sont les opérateurs naturels associés, voir Definition 3.2.17. Les propriétés supplémentaires vérifiées par ces partitions non-croisées décorées par les éléments de Γ, $p \in NC_\Gamma$ proviennent des relations contenues dans le noyau du morphisme $\rho : C(H_N^{\infty+})^{*p} \to C(H_N^+(\Gamma))$.

On obtient alors les règles de fusion des produits en couronnes libres $\widehat{\Gamma} \wr_* S_N^+$ pour tout groupe Γ (sans nécessairement supposer que Γ est finiment engendré). Précisément, on obtient le Théorème 3.2.25 que l'on peut résumer comme suit : les coreprésentations irréductibles de $H_N^+(\Gamma)$ peuvent être indexées par les mots $(g_1, \ldots, g_k) \in \langle \Gamma \rangle$ avec l'involution

$\overline{(g_1,\dots,g_k)} = (g_k^{-1},\dots,g_1^{-1})$ et les règles de fusion:

$$\begin{aligned}
&(g_1,\dots,g_k)\otimes(h_1,\dots,h_l)\\
&=(g_1,\dots,g_{k-1},g_k,h_1,h_2,\dots,h_l)\oplus(g_1,\dots,g_{k-1},g_kh_1,h_2,\dots,h_l)\\
&\oplus\delta_{g_kh_1,e}(g_1,\dots,g_{k-1})\otimes(h_2,\dots,h_l).
\end{aligned}$$

Dans les Sous-sections 3.3.1 et 3.3.2, on exploite la connaissance de ces règles de fusion pour étudier les algèbres d'opérateurs des produits en couronnes libres $\Gamma \wr_* S_N^+$. Premièrement, on montre que $C_r(H_N^+(\Gamma))$ est simple avec une trace unique, ce qui implique en particulier que $L^\infty(H_N^+(\Gamma))$ est un facteur de type II_1, voir Theorem 3.3.5 et son corollaire. Pour cela on utilise des méthodes proches de celles de Powers pour la preuve de la simplicité de la C^*-algèbre réduite du groupe libre. On s'inspire notamment de la preuve de Banica pour la simplicité de $C_r(U_N^+)$, [Ban97]. Il s'agit de remplacer, dans les méthodes classiques de Powers, les partitions du groupe libre par une partition des irréductibles de $H_N^+(\Gamma)$. Les actions par automorphismes intérieurs sont remplacées par l'action des caractères via l'application adjointe. Celle-ci produit des applications complètement positives $T_i : C_r(H_N^+(\Gamma)) \to C_r(H_N^+(\Gamma))$

$$x \mapsto \sum_k a_k x a_k^*$$

avec des coefficients a_k qui proviennent de coreprésentations irréductibles de $H_N^+(\Gamma)$. Suivant les techniques de Powers, on fixe trois coreprésentations irréductibles distinctes r_i et l'on obtient alors une application unifère complètement positive $T = C\sum_i T_i$, où C est le facteur de renormalisation. Cette application est contractante sur un certain sous-espace $\mathscr{S}$ de $C_r(H_N^+(\Gamma))$, engendré par les coefficients de coreprésentations irréductibles indexées par des mots commençant et se terminant par une lettre différente de e_Γ. En partant, s'il existe, d'un élément $0 \neq x \in J \cap \mathscr{S}$ où J est un idéal bilatère non nul $J \lhd C_r(H_N^+(\Gamma))$, on peut alors grâce à la forme des applications T_i ci-dessus, itérer l'application T et obtenir un élément de J proche de l'unité :

$$||1 - T^{(m)}(x)||_r = ||T^{(m)}(1-x)||_r < 1.$$

On peut alors conclure que $J = C_r(H_N^+(\Gamma))$. Le problème est donc de produire un tel élément $x \in J \cap \mathscr{S}$. La différence majeure avec la méthode de Banica dans le cas du groupe quantique U_N^+ provient du fait que $C(H_N^+(\Gamma))$ contient une copie de $C(S_N^+)$. L'ensemble des coefficients des coreprésentations indexées par les mots $e_\Gamma^k = (e_\Gamma,\dots,e_\Gamma)$ constitués uniquement de la lettre e_Γ, produisent en effet une sous-$*$-algèbre $\mathscr{C}$ isomorphe à $Pol(S_N^+)$. Les règles de fusion dans $H_N^+(\Gamma)$ montrent que si le support de $x \in J$ contient une coreprésentation indexée par un mot e^k, on ne pourra pas nécessairement produire un élément de $J \cap \mathscr{S}$ "en conjuguant" x. On remédie à cela en utilisant l'espérance conditionnelle $P : C_r(H_N^+(\Gamma)) \to C_r(S_N^+)$ qui produit alors un antécédent de $1 = E(x)$ grâce à la simplicité de $C_r(S_N^+)$. On obtient alors une décomposition $x = 1 - z$ où $z \in \mathscr{S}$. On peut finalement appliquer les méthodes de Banica à $z \in \mathscr{S}$ et conclure. La preuve de l'unicité de la trace suit des raisonnements similaires, voir Theorem 3.3.5.

Enfin, on montre la plénitude du facteur $L^\infty(H_N^+(\Gamma))$ de type II_1, Theorem 3.3.10. La démonstration repose sur une adaptation de la méthode des $14 - \epsilon$. Vaes l'a adapté par exemple pour montrer la plénitude de $L^\infty(U_N^+)$. A nouveau, on doit se ramener au cas où les éléments avec lesquels on travaille sont dans l'espace $\mathscr{S}$ ci-dessus et utiliser la plénitude de $L^\infty(S_N^+)$.

Dans la Section 1.3 du Chapitre 1, on motivera le fait que ces propriétés de plénitude et donc de non-injectivité pour les produits en couronnes libres sont liées à des résultats de structures des algèbres de von Neumann de groupes (quantiques) obtenus récemment.

Chapter 1

Preliminaries

1.1 Quantum groups

One motivation for the construction of quantum groups was the generalization of Pontrjagin duality to non-abelian locally compact groups: if G is an abelian locally compact group then the set of characters $\widehat{G}$ is an abelian locally compact group again and $\widehat{\widehat{G}} \simeq G$. Of course if G is no longer abelian, one can not expect that $\widehat{\widehat{G}} \simeq G$ and then one has to look for a larger category, the one of quantum groups, that includes locally compact groups and their (generalized) duals. In [VK74] and [ES75], the authors defined the notion of a Kac algebra A in the setting of von Neumann algebras, of the dual Kac algebra $\widehat{A}$ and proved that $\widehat{\widehat{A}} \simeq A$. Kac algebras are endowed with the same structural maps as Hopf-algebras (coproduct, antipode, counit). A C^*-algebraic theory and analogue results were proved in this setting, see [EV93]. These algebras together with their structural maps are examples of quantum groups.

However, in [Wor87b] the author constructed a C^*-algebra, the so called twisted $SU_q(2)$ quantum groups, with a coproduct but with an unbounded antipode that is not a $*$-anti-automorphism as in the case of Kac algebras. So the category of Kac algebras appeared not large enough to contain all interesting examples of quantum groups. In [Wor87a], [Wor98], the author introduced a general theory of compact quantum groups in the setting of C^*-algebras. This construction includes also the Drinfeld-Jimbo type quantum groups. Under minimal assumptions, the existence and uniqueness of a Haar state could be proved and a Peter-Weyl (co-)representation theory of compact quantum groups could be developed, very close to the one for (classical) compact groups. Moreover, as in the classical case the dual $\widehat{\mathbb{G}}$ of a compact quantum group $\mathbb{G}$ is a discrete quantum group i.e. the underlying C^*-algebra is direct sum of matrix algebras. The matrix algebras arise from the representation theory of $\mathbb{G}$, the irreducible $\mathbb{G}$-representations being finite dimensional.

In [BS93], the authors used multiplicative unitaries to encode simultaneously a compact quantum group and its discrete dual. These multiplicative unitaries are also a key tool in the general theory of locally compact quantum groups in the setting of C^*/von Neumann algebras (see e.g. [KV00]) giving a complete answer to the generalization of Pontrjagin duality mentioned above.

1.1.1 Compact quantum groups

In this subsection, we give basic definitions, examples, results and constructions relative to compact quantum groups that we will need in the sequel. One can refer to [Wor98], [MVD98] and to [Tim] for more details. We denote by $A \otimes B$ the minimal tensor product of two C^*-algebras faithfully represented on Hilbert spaces (π_A, H_A), (π_B, H_B) that is the completion of the algebraic tensor product $A \odot B$ with respect to the C^*-norm induced by the representation $\pi_A \otimes \pi_B$ on $H_A \otimes H_B$.

Definition 1.1.1. *A compact quantum group $\mathbb{G}$ is a pair $\mathbb{G} = (C(\mathbb{G}), \Delta)$ where $C(\mathbb{G})$ is a Woronowicz-C^*-algebra, that is a unital C^*-algebra together with a $*$-homomorphism $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ such that*

1. $\Delta(C(\mathbb{G}))(1 \otimes C(\mathbb{G}))$ *and* $\Delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes 1)$ *are linearly dense in* $C(\mathbb{G}) \otimes C(\mathbb{G})$,
2. $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$.

Example 1.1.2. *Two basic examples arise from groups:*

- *Let G be a compact group, consider $C(G)$ the algebra of complex functions on G. Then $(C(G), \Delta)$ is a compact quantum group with $\Delta : C(G) \to C(G) \otimes C(G) \simeq C(G \times G), \Delta(f)(x, y) = f(xy)$. Notice that these examples are commutative.*
- *Let Γ be a discrete group, and consider the reduced group C^*-algebra $C_r^*(\Gamma)$, then $(C_r^*(\Gamma), \Delta)$ is a compact quantum group with $\Delta(g) = g \otimes g$, $\forall g \in \Gamma$. Notice that these examples are cocommutative, that is $\Sigma \circ \Delta = \Delta$ if Σ denotes the flip map in $C^*(\Gamma) \otimes C^*(\Gamma)$.*

The minimal assumptions of Definition 1.1.1 above allow to prove the following theorem:

Theorem 1.1.3. *Let $\mathbb{G}$ be a compact quantum group. There exists a unique state, called Haar state, $h : C(\mathbb{G}) \to \mathbb{C}$ satisfying the bi-invariance relations:*

$$(h \otimes id) \circ \Delta(\cdot) = h(\cdot)1 = (id \otimes h) \circ \Delta(\cdot)$$

The Haar state h is not a trace in general. When h is a trace, we say that the compact quantum group is of Kac type. However, this state is always a KMS state.

If G is a compact group, the relations above are the analogue of the left invariance relation satisfied by the Haar measure: $\int_G f(xg)d_g = \int_G f(g)d_g = \int_G f(gx)dg$ for all $x \in G$ and all $f : G \to \mathbb{C}$.

We can recall here the construction of the reduced C^*-algebra associated to $\mathbb{G}$. Let λ_h be the GNS-representation of the Haar state h (also called left regular representation of $C(\mathbb{G})$), $\lambda_h : C(\mathbb{G}) \to B(L^2(\mathbb{G}))$ with GNS space denoted by $L^2(\mathbb{G})$. We recall that this construction comes with a GNS map $\Lambda : C(\mathbb{G}) \to L^2(\mathbb{G})$ with dense image and such that

$$\langle \Lambda(a), \Lambda(b) \rangle = h(a^*b) \text{ and } \lambda_h(a)\Lambda(b) = \Lambda(ab),\ \forall a, b \in C(\mathbb{G}).$$

There exists also a cyclic vector $\xi_0 = \Lambda(1)$, that is a unit vector such that

$$\Lambda(a) = \lambda_h(a)\xi_0 \text{ and } h(a) = \langle \xi_0, \lambda_h(a)\xi_0 \rangle,\ \forall a \in C(\mathbb{G}).$$

The reduced C^*-algebra associated with $\mathbb{G}$ is defined and denoted as follows:

$$C_r(\mathbb{G}) := \lambda_h(C(\mathbb{G})) \simeq C(\mathbb{G})/ker(\lambda_h).$$

One can define a canonical coproduct Δ_r and Haar state h_r on $C_r(\mathbb{G})$ with the formulas

$$\Delta_r \circ \lambda_h = (\lambda_h \otimes \lambda_h)\Delta,\ h_r \circ \lambda_h = h$$

and one can show that $\mathbb{G}_r := (C_r(\mathbb{G}), \Delta_r)$ is a compact quantum group on which h_r is faithful. We will denote h_r simply by h in the sequel.

The compact quantum group $\mathbb{G}$ also admits a maximal (or universal) version $\mathbb{G}_u$. The underlying C^*-algebra $C(\mathbb{G}_u)$ is the completion of a certain dense $*$-subalgebra $Pol(\mathbb{G}) \subset C(\mathbb{G})$ with respect to the following norm:

$$||a||_u = \sup_{\pi} ||\pi(a)||,$$

where π runs over all the unital $*$-representations of $Pol(\mathbb{G})$. The algebra $Pol(\mathbb{G})$ is an essential object when one studies the properties of a compact quantum group $\mathbb{G}$, we give a precise definition of this algebra below, see (1.2).

Before giving the results of Peter-Weyl's (co-)representation theory for compact quantum group, we recall that if $B(H)$ denotes the algebra of bounded operators on a Hilbert space then $B(H)$ is the multiplier algebra of $K(H)$ the C^*-algebra of compact operators (i.e. $B(H)$ is the largest C^*-algebra containing $K(H)$ as an *essential* ideal).

Definition 1.1.4. *Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group. A representation of $\mathbb{G}$ (or corepresentation of $C(\mathbb{G})$) on a Hilbert space H is an element in the multiplier algebra $M(C(\mathbb{G}) \otimes K(H))$ such that*

$$(\Delta \otimes id)(v) = v_{(13)}v_{(23)}, \tag{1.1}$$

where $v_{(13)} = (1 \otimes \Sigma)(v \otimes 1)$ and $v_{(23)} = 1 \otimes v$ and Σ is the flip map. If v is unitary then the corepresentation is said to be unitary.

If G is a compact group and $\pi : G \to B(H)$ is a strongly continuous unitary representation then π is also continuous for the strict topology $\pi : G \to M(K(H))$. Then one can see $\pi \in M(C(G) \otimes K(H))$ and one can identify elements in $M(C(G \times G) \otimes K(H))$ with strictly continuous $B(H)$-valued functions on $G \times G$. We get

$$\pi_{(13)}(g, h) = \pi(g),\ \pi_{(23)}(g, h) = \pi(h),\ \forall g, h \in G$$

and since

$$(\Delta \otimes id)(\pi)(g, h) = \pi(gh),$$

the relation (1.1) simply translates into

$$\pi(gh) = \pi(g)\pi(h).$$

We will mainly deal with unitary finite dimensional corepresentations. They are unitary matrices $U \in M_N(C(\mathbb{G})) \simeq C(\mathbb{G}) \otimes M_N(\mathbb{C})$ with entries in $C(\mathbb{G})$ satisfying

$$\Delta(U_{ij}) = \sum_{k=1}^{N} U_{ik} \otimes U_{kj}.$$

This is simply the translation of the relation (2.2) to this situation. We say that $T \in M_{N_u,N_v}(\mathbb{C})$ is an intertwiner between two corepresentations $u \in M_{N_u}(C(\mathbb{G})), v \in M_{N_u}(C(\mathbb{G}))$ if T is a matrix such that $v(1 \otimes T) = (1 \otimes T)u$. We write $T \in Hom_{\mathbb{G}}(u, v)$. We say that u and v are equivalent, and we write $u \sim v$, if T is invertible. We say that a finite dimensional corepresentation u is irreducible if $Hom(u, u) = \mathbb{C}id$.

All these definitions have of course non finite-dimensional counterparts. However, in the next chapters and especially in Chapter 3, when we compute the fusion rules of the free wreath products $\widehat{\Gamma} \wr_* S_N^+$, we will only have to deal with finite-dimensional corepresentations thanks to the following theorem:

Theorem 1.1.5. *Every irreducible corepresentation of a compact quantum group is finite-dimensional. Furthermore, every unitary corepresentation is unitarily equivalent to a direct sum of irreducible corepresentations.*

We will denote by $Irr(\mathbb{G})$ the set of equivalence classes of irreducible corepresentations of $C(\mathbb{G})$. For each $\alpha \in Irr(\mathbb{G})$, the previous theorem allows us to choose a unitary representative corepresentation U^α. We will denote by H_α the representation space of U^α.

We will denote by

$$Pol(\mathbb{G}) \subset C(\mathbb{G}) \tag{1.2}$$

the subspace generated by the coefficients $(id \otimes \phi)(U^\alpha), \phi \in B(H_\alpha)^*$ of irreducible corepresentations $U^\alpha, \alpha \in Irr(\mathbb{G})$. It is a remarkable fact that this sub-algebra is dense in $C(\mathbb{G})$. The irreducible characters of $\mathbb{G}$ are the traces $\chi_\alpha = \sum_{i=1}^{d_\alpha} U_{ii}^\alpha$.

If U^α is an N-dimensional unitary corepresentation of $C(\mathbb{G})$, $U^\alpha = (U_{ij}^\alpha) \in M_N(C(\mathbb{G}))$, $\overline{U^\alpha} := ((U_{ij}^\alpha)^*)$ is also a corepresentation called the conjugate of U^α. However it is not necessarily unitary anymore but there exists a unique positive definite $Q_\alpha \in M_N(\mathbb{C})$ with $Tr(Q_\alpha) = Tr(Q_\alpha^{-1})$ such that $Q_\alpha^{1/2}\overline{U^\alpha}Q_\alpha^{-1/2}$ is unitary. The number $Tr(Q_\alpha)$ is called the quantum dimension of U^α, $dim_q(U^\alpha)$. If $Q_\alpha = id$ then the quantum dimension is the usual dimension $N = dim(U^\alpha)$. Notice that, in this thesis, we will mainly deal with Kac type compact quantum groups $\mathbb{G}$. $\mathbb{G}$ is of Kac type if and only if $Q_\alpha = id$ for all $\alpha \in Irr(\mathbb{G})$.

Furthremore, we have the following "Schur's" orthogonality relations for any $U^\alpha, \alpha \in Irr(\mathbb{G})$:

$$h((U_{ji}^\alpha)^* U_{kl}^\beta) = \delta_{\alpha,\beta}\delta_{il}\frac{(Q_\alpha^{-1})_{jk}}{Tr(Q_\alpha)},\ h(U_{ij}^\alpha (U_{kl}^\beta)^*) = \delta_{\alpha,\beta}\delta_{ik}\frac{(Q_\alpha)_{jl}}{Tr(Q_\alpha)},$$

One can prove that $Pol(\mathbb{G})$ is a Hopf $*$-algebra. That is $Pol(\mathbb{G})$ is a unital $*$-algebra endowed with the following structural maps:

- a $*$-homomorphism called coproduct : $\delta : Pol(\mathbb{G}) \to Pol(\mathbb{G}) \odot Pol(\mathbb{G})$ satisfying $(id \otimes \delta) \circ \delta = (\delta \otimes id) \circ \delta$,
- a $*$-homorphism $\epsilon : Pol(\mathbb{G}) \to \mathbb{C}$ called counit satifying $(\epsilon \otimes id) \circ \delta = id = (id \otimes \epsilon) \circ \delta$,
- a linear map called antipode : $\kappa : Pol(\mathbb{G}) \to Pol(\mathbb{G})$ satisfying $m \circ (\kappa \otimes id) \circ \delta = \eta \circ \epsilon = m \circ (id \otimes \kappa) \circ \delta$,

where $\eta : \mathbb{C} \to Pol(\mathbb{G})$ is the unit map $\lambda \mapsto \lambda 1_{C(\mathbb{G})}$ and $m : Pol(\mathbb{G}) \otimes Pol(\mathbb{G}) \to Pol(\mathbb{G})$ is the multiplication map $m(a \otimes b) = ab$. Of course δ is the restriction of $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ to $Pol(\mathbb{G})$ and we will denote δ by Δ in the sequel.

The antipode satisfies: $\kappa \circ * \circ \kappa \circ * = id$. Thus κ is bijective and actually it is a antiautomorphism on $Pol(\mathbb{G})$. We recall that a compact quantum group is of Kac type if and only if the Haar state h is a trace. This is also equivalent to the fact the $\kappa^2 = id$.

Starting with a family of (irreducible) corepresentations, one can construct new corepresentations. Of course one can consider the direct sum of two corepresentations. One can also define the tensor product of corepresentations thanks to the product in $C(\mathbb{G})$ as follows:

Definition 1.1.6. *Let U and V be unitary $\mathbb{G}$-corepresentations on certain Hilbert spaces H_U and H_V. Then the tensor product of U and V is defined as follows*

$$U \otimes V := U_{(12)} V_{(13)} \in M(C(\mathbb{G}) \otimes K(H_U \otimes H_V)).$$

In view of Theorem 1.1.5, it is clear that an essential question when one studies compact quantum groups is to describe their irreducible corepresentations and the fusion rules binding them. We recall:

Definition 1.1.7. *Let $\mathbb{G}$ be a compact quantum group with Woronowicz C^*-algebra $C(\mathbb{G})$. The fusion semiring $R^+(C(\mathbb{G}))$ is the set of equivalence class of finite dimensional corepresentations of $C(\mathbb{G})$, endowed with the direct sum of (classes) of corepresentations and the tensor product of (classes) of corepresentations.*

The fusion semiring $R^+(C(\mathbb{G}))$ encodes the relations of the form

$$\alpha \otimes \beta = \bigoplus_{i \in I} \gamma_i$$

with $\alpha, \beta, \gamma_i \in Irr(\mathbb{G})$ and some finite set I. These formulas then describe the splitting of a tensor product of irreducible corepresentations into a sum of irreducible corepresentations and are called fusion rules for the irreducible corepresentations of $C(\mathbb{G})$. We denote $\gamma \subset \alpha \otimes \beta$ if the irreducible corepresentation γ appears in the decomposition of the tensor product $\alpha \otimes \beta$. Such an inclusion gives rise to linear maps $T : H_\gamma \to H_\alpha \otimes H_\beta$ and thus to morphisms $(1 \otimes T) \in Hom(\gamma; \alpha \otimes \beta)$. We will call multiplicity of $\gamma \subset \alpha \otimes \beta$ the dimension of the space $Hom(\gamma; \alpha \otimes \beta)$.

We will give examples of compact quantum groups and the fusion rules binding their irreducible corepresentations in the Subsection 1.1.4.

The knowledge of these fusion rules is a key tool if one investigates the operator algebraic properties satisfied by the C^*/von Neumann algebras associated to quantum groups (see Chapters 2, 3). We now recall briefly the construction of the duals of compact quantum groups.

1.1.2 Discrete quantum groups

A discrete quantum group is the dual of a compact one. Starting with a complete set $\{U^\alpha : \alpha \in Irr(\mathbb{G})\}$ of pairwise non-equivalent $\mathbb{G}$-irreducible corepresentations it is easy to construct the dual quantum group $\widehat{\mathbb{G}}$. The dual $\widehat{\mathbb{G}}$ of $\mathbb{G}$ is the pair $(B, \widehat{\Delta})$ where B is the completion (with respect to a naturally arising C^*-norm, see below) of the subspace $B_0 \subset C(\mathbb{G})^*$ generated by the linear functionals on $C(\mathbb{G})$ defined by $x \mapsto h(ax)$ for $a \in Pol(\mathbb{G})$. One shows that B_0 is a multiplier Hopf $*$-algebra with structural maps obtained by dialyzing the ones of $C(\mathbb{G})$. Furthermore, B_0 is a direct sum of matrix algebras, and the completion of B_0 with respect to the unique C^*-norm on B_0 then gives the C^*-algebra B. One can refer to [MVD98] for more details.

Standard notations are the following ones. If $\mathbb{G}$ be a compact quantum group, the underlying C^*-algebra B of $\widehat{\mathbb{G}}$ is isomorphic with:

$$c_0(\widehat{\mathbb{G}}) = c_0 - \bigoplus_{\alpha \in Irr(\mathbb{G})} B(H_\alpha)$$

and the coproduct $\widehat{\Delta}$ implemented by natural unitary $W \in M(C(\mathbb{G}) \otimes c_0(\widehat{\mathbb{G}}))$,

$$W = \bigoplus_{\alpha \in Irr(\mathbb{G})} U^\alpha,$$

$$\widehat{\Delta} : c_0(\widehat{\mathbb{G}}) \to c_0(\widehat{\mathbb{G}}) \otimes c_0(\widehat{\mathbb{G}}),\ (id \otimes \widehat{\Delta})(W) = W_{(12)} W_{(13)}.$$

The unitary W is a multiplicative unitary in the sense of [BS93] that is W satisfies the pentagonal equation:

$$W_{(12)} W_{(13)} W_{(23)} = W_{(23)} W_{(12)}.$$

Notice that $\widehat{\Delta}$ is characterized by $\widehat{\Delta}(x)T = Tx$ for $x \in B(H_\gamma)$ and $T \in Hom(\gamma, \alpha \otimes \beta)$. One can also associate a von Neumann algebra to the discrete quantum group $\widehat{\mathbb{G}}$:

$$l^\infty(\widehat{\mathbb{G}}) = l^\infty - \bigoplus_{\alpha \in Irr(\mathbb{G})}^{\infty} B(H_\alpha).$$

We denote by p_α the minimal central projection in $l^\infty(\widehat{\mathbb{G}})$ corresponding to the identity of $B(H_\alpha)$. One can then define weights ϕ, ψ on $c_0(\widehat{\mathbb{G}})$ as follows:

$$\phi(x) = \sum_{\alpha \in Irr(\mathbb{G})} dim_q(\alpha) Tr(Q_\alpha p_\alpha x),$$
$$\psi(x) = \sum_{\alpha \in Irr(\mathbb{G})} dim_q(\alpha) Tr(Q_\alpha^{-1} p_\alpha x),$$

with Q_α defined in the previous subsection. They are left and right invariant under $\widehat{\Delta}$ respectively. However, they are not states in general (in fact $c_0(\widehat{\mathbb{G}})$ is not unital unless it is finite dimensional) and they do not coincide either. If they are equal, we say that $\widehat{\mathbb{G}}$ is unimodular. This is equivalent to the fact that $\mathbb{G}$ is of Kac type.

1.1.3 Regular and adjoint representations

Let $\mathbb{G}$ be a Kac type compact quantum group with Haar trace h. Recall that we denote by $\lambda_h : C(\mathbb{G}) \to B(L^2(\mathbb{G}))$ the regular representation of $\mathbb{G}$, that is the GNS construction $(\lambda_h, \Lambda_h, L^2(\mathbb{G}))$ associated with the Haar state h:

$$\lambda_h(a)\Lambda_h(b) = \Lambda_h(ab) \in L^2(\mathbb{G}), \forall a, b \in C(\mathbb{G}).$$

In the Kac type case, the right regular representation $\rho_h : C(\mathbb{G}) \to B(L^2(\mathbb{G}, h))$ is given by

$$\rho_h(x)\Lambda_h(y) = \Lambda_h(y\kappa(x)).$$

In the non Kac type case, one has to convolve by Woronowicz's characters inside the antipode. The right regular representation commutes with λ_h and if $\mathbb{G}$ is full, that is if the underlying Woronowicz C^*-algebra is maximal, one can consider the adjoint representation

$$ad := (\lambda_h, \rho_h) \circ \Delta : C_u(\mathbb{G}) \to B(L^2(\mathbb{G}, h)).$$

In the Kac case, the irreducible characters act as follows

$$ad(\chi_r)(z) = \sum_{ij} r_{ij} z r_{ij}^*.$$

Notice that the map $z \mapsto ad(\chi_r)(z)$ is completely positive for all $r \in Irr(\mathbb{G})$. We will use these maps in Chapter 3.

1.1.4 Examples: free compact quantum groups

We will deal with many classical examples of compact quantum groups in this thesis. Some of the techniques we will use are adapted from existing ones relative to the following examples.

Free unitary quantum groups The following family of examples was introduced by Wang in [Wan93].

Definition 1.1.8. *Let $N \geq 2$. The free unitary quantum group is the pair $U_N^+ = (C(U_N^+), \Delta)$ where $C(U_N^+)$ is the universal C^*-algebra generated by N^2 elements U_{ij} such that the matrices U and $\bar{U}$ are unitary and*

$$\Delta : C(U_N^+) \to C(U_N^+) \otimes C(U_N^+)$$

is the unital $*$-homomorphism such that for all $i, j \in \{1, \ldots, N\}$

$$\Delta(U_{ij}) = \sum_{k=1}^{N} U_{ik} \otimes U_{kj}.$$

For any matrix $F \in GL_N(\mathbb{C})$, one can define a compact quantum group $U_F^+ = (A_u(F), \Delta)$ where $A_u(F)$ is the universal C^*-algebra generated by N^2 elements U'_{ij} such that $U' = (U'_{ij})$ and $F\bar{U}'F^{-1}$ are unitary and Δ is a unital $*$-homomorphism

$$\Delta : A_u(F) \to A_u(F) \otimes A_u(F)$$

such that

$$\Delta(U'_{ij}) = \sum_{k=1}^{N} U'_{ik} \otimes U'_{kj}$$

for all $\forall i, j \in \{1, \ldots, N\}$. In this thesis, we will mainly deal with $A_u(I_N)$ that is to say with the free unitary quantum groups U_N^+.

Banica has computed the fusion rules of U_N^+ in [Ban97]: we recall one of the main results of this paper in the next theorem. Let us first fix some notations. We denote by $\mathbb{N} * \mathbb{N}$ the free product of the monoid $\mathbb{N}$ with itself and by $\alpha, \beta \in \mathbb{N} * \mathbb{N}$ two canonical generators and by $\emptyset$ the empty word. We will consider the unique anti-multiplicative involution on $\mathbb{N} * \mathbb{N}$ defined by $\bar{\emptyset} = \emptyset, \bar{\alpha} = \beta$ and $\bar{\beta} = \alpha$.

Theorem 1.1.9. *There exists a family of pairwise inequivalent irreducible corepresentations $(U_x)_{x \in \mathbb{N} * \mathbb{N}}$ of U_N^+ such that*

$$U_\emptyset = 1_{C(U_N^+)},\ U_\alpha = U,\ U_\beta = \bar{U},\ \overline{U_x} = U_{\bar{x}}\ \forall x \in \mathbb{N} * \mathbb{N},$$

and for all $x, y \in \mathbb{N} * \mathbb{N}$

$$U_x \otimes U_y = \sum_{\substack{x=ac \\ y=\bar{c}b}} U_{ab}.$$

Furthermore, every irreducible representation of U_N^+ *is equivalent to a corepresentation* U_x *for some* $x \in \mathbb{N} * \mathbb{N}$.

Free orthogonal quantum groups The orthogonal case was also introduced by Wang in [Wan93].

Definition 1.1.10. *Let* $N \geq 2$. *The free orthogonal quantum group is the pair* $O_N^+ = (C(O_N^+), \Delta)$ *where* $C(O_N^+)$ *is the universal* C^**-algebra generated by* N^2 *self adjoint elements* V_{ij} *such that the matrix* $V = (V_{ij})$ *is unitary and*

$$\Delta : C(O_N^+) \to C(O_N^+) \otimes C(O_N^+)$$

is a unital $*$*-homomorphism such that for all* $i, j \in \{1, \dots, N\}$

$$\Delta(V_{ij}) = \sum_{k=1}^{N} V_{ik} \otimes V_{kj}.$$

For any matrix $F \in GL_N(\mathbb{C})$, one can define a compact quantum group $O_F^+ = (A_o(F), \Delta)$ where $A_o(F)$ is the universal C^*-algebra generated by N^2 elements V'_{ij} such that $V' = (V'_{ij})$ is unitary and $F\bar{V}'F^{-1} = V'$ and Δ is a unital $*$-homomorphism

$$\Delta : A_o(F) \to A_o(F) \otimes A_o(F)$$

such that

$$\Delta(V'_{ij}) = \sum_{k=1}^{N} V'_{ik} \otimes V'_{kj}$$

$\forall i, j \in \{1, \dots, N\}$. In this thesis, we will mainly deal with $A_o(I_N)$ that is to say with the free orthogonal quantum groups O_N^+.

Banica has computed the fusion rules for the orthogonal quantum groups in [Ban96]. We recall the main theorem of this paper in the following theorem:

Theorem 1.1.11. *There exists a family of pairwise inequivalent irreducible corepresentations* $(V^n)_{n \in \mathbb{N}}$ *of* O_N^+ *such that for all* $n \in \mathbb{N}$

$$V^0 = 1_{C(O_N^+)}, \quad V^1 = V, \ \overline{V^n} \sim V^n \ \forall n \in \mathbb{N},$$

and for all $r, s \in \mathbb{N}$

$$V^r \otimes V^s = \bigoplus_{l=0}^{\min\{r,s\}} V^{r+s-2l}.$$

Furthermore, every irreducible representation of O_N^+ is equivalent to a corepresentation V_n for some $n \in \mathbb{N}$.

The dimensions of the corepresentations V^n are given for all $N \geq 3$ by:

$$d_n := \dim V^n = \frac{q(N)^{n+1} - q(N)^{-n-1}}{q(N) - q(N)^{-1}} \; \forall n \in \mathbb{N},$$

where $q(N) = \frac{N+\sqrt{N^2-4}}{2}$.

If $N = 2$, we simply have $d_n = n + 1$ for all $n \in \mathbb{N}$.

The irreducible characters $\chi_n := \sum_{i=1}^{d_n} V_{ii}^n$ satisfy the recursive relations:

$$\chi_1 \chi_n = \chi_{n+1} + \chi_{n-1} \; \forall n \geq 1$$

One can prove that the "complexification" of O_N^+ is U_N^+: we have an embedding at the level of the reduced C^*-algebras $C_r(U_N^+) \subset C(\mathbb{T}) *_r C_r(O_N^+)$ given by

$$\pi_{h_{U_N^+}}(U_{ij}) \mapsto z\pi_{h_{O_N^+}}(V_{ij})$$

where $z = id_{\mathbb{T}}$ designates a generator of $C(\mathbb{T})$, $\pi_{h_{U_N^+}}$ (resp. $\pi_{h_{O_N^+}}$) is the GNS representation associated to the Haar state on U_N^+ (resp. O_N^+) and the reduced free product is taken with respect to these Haar states (see e.g. [BC07, Theorem 9.2]). When we discuss Haagerup property and recall the results proved by Brannan, this fact will be important.

Quantum permutation groups Another important and well studied family of free compact quantum groups was also introduced by Wang in [Wan98].

Definition 1.1.12. *Let $N \geq 2$. The quantum permutation group is the pair $S_N^+ = (C(S_N^+), \Delta)$ where $C(S_N^+)$ is the universal C^*-algebra generated by N^2 elements v_{ij} such that the matrix $v = (v_{ij})$ is magic unitary that is to say its entries are pairwise orthogonal projections which sum up to one on each row and column and*

$$\Delta : C(S_N^+) \to C(S_N^+) \otimes C(S_N^+)$$

is a unital $*$-homomorphism such that for all $i, j \in \{1, \ldots, N\}$

$$\Delta(v_{ij}) = \sum_{k=1}^{N} v_{ik} \otimes v_{kj}.$$

It is possible to define a (universal) quantum automorphism group $\mathbb{G}_{aut}(B, \psi)$ for any finite dimensional C^*-algebra equipped with a faithful state ψ in terms of right action of compact quantum group on B (see [Wan98] and [Ban02]). The case of S_N^+ coincides with the situation where $B = C(\{1, \ldots, N\}) = \mathbb{C}^N$ and ψ is the uniform probability measure on $\{1, \ldots, N\}$. In this thesis, we will mainly deal with the case of the quantum permutation group S_N^+.

Banica computed the fusion rules for the quantum permutation groups in [Ban99b], we recall the results in the following theorem:

Theorem 1.1.13. *There exists a family of pairwise inequivalent irreducible corepresentations $(v^n)_{n\in\mathbb{N}}$ of S_N^+ such that for all $n \in \mathbb{N}$*

$$v^0 = 1_{C(S_N^+)},\ v = 1 \oplus v^1,\ \overline{v^n} \simeq v^n\ \forall n \in \mathbb{N}$$

and for all $r, s \in \mathbb{N}$

$$v^r \otimes v^s = \bigoplus_{l=0}^{2\min\{r,s\}} v^{r+s-l}.$$

Furthermore, every irreducible representation of S_N^+ is equivalent to a corepresentation v_n for some $n \in \mathbb{N}$.

The dimensions d_n, of the corepresentations v^n are given recursively by :

$$d_0 = 1, d_1 := N - 1 \ \text{ and } d_1 d_n = d_{n+1} + d_n + d_{n-1}\ \forall n \in \mathbb{N}.$$

The irreducible characters $\chi_n := \sum_{i=1}^{d_n} v_{ii}^n$ satisfy the recursive relations:

$$\chi_1\chi_n = \chi_{n+1} + \chi_n + \chi_{n-1}\ \forall n \geq 1.$$

Free wreath product quantum groups In [Bic04], Bichon introduced new examples of compact quantum groups. If A is a unital C^*-algebra, we denote by ν_i the canonical homomorphism $\nu_i : A \to A^{*N}$ sending A to the i-th copy of A in the full free product A^{*N}.

Definition 1.1.14. *Let A be a Woronowicz-C^*-algebra and $N \geq 2$. The free wreath product of A by the quantum permutation group S_N^+ is the quotient of the C^*-algebra $A^{*N} * C(S_N^+)$ by the two-sided ideal generated by the elements*

$$\nu_k(a)v_{ki} - v_{ki}\nu_k(a), \quad 1 \leq i, k \leq N, \quad a \in A.$$

It is denoted by $A *_w C(S_N^+)$.

Theorem 1.1.15. *Let A be a Woronowicz-C^*-algebra, then free wreath product $A *_w C(S_N^+)$ admits a Woronowicz-C^*-algebra structure: $\forall a \in A, \forall\ 1 \leq i, j \leq N$,*

$$\Delta(v_{ij}) = \sum_{k=1}^{N} v_{ik} \otimes v_{kj} \ ; \ \Delta(\nu_i(a)) = \sum_{k=1}^{N} \nu_i(a_{(1)})v_{ik} \otimes \nu_k(a_{(2)})$$

$$\epsilon(v_{ij}) = \delta_{ij} \ ; \ \epsilon(\nu_i(a)) = \epsilon_A(a) \ ; \ S(v_{ij}) = v_{ji} \ ; \ S(\nu_i(a)) = \sum_{k=1}^{N} \nu_k(S_A(a))v_{ki}.$$

$$v_{ij}^* = v_{ij} \ ; \ \nu_i(a)^* = \nu_i(a^*).$$

Moreover, if $\mathbb{G}$ is a full compact quantum group, then $\mathbb{G} \wr_* S_N^+ = (A *_w C(S_N^+), \Delta)$ is also a full compact quantum group.

Special cases of these free wreath products were studied:

Example 1.1.16. *Let Γ be a (discrete) group, $N \geq 2$. Let $C(H_N^+(\Gamma))$ be the universal C^*-algebra with generators $a_{ij}(g), 1 \leq i, j \leq N, g \in \Gamma$ together with the following relations:*

$$a_{ij}(g)a_{ik}(h) = \delta_{jk}a_{ij}(gh) \quad ; \quad a_{ji}(g)a_{ki}(h) = \delta_{jk}a_{ji}(gh) \quad ; \quad \sum_{l=1}^{N} a_{il}(e) = 1 = \sum_{l=1}^{N} a_{li}(e),$$

and involution $a_{ij}(g)^* = a_{ij}(g^{-1})$. Then $H_N^+(\Gamma) := (C(H_N(\Gamma)), \Delta)$ is a compact quantum group with:

$$\Delta(a_{ij}(g)) = \sum_{k=1}^{N} a_{ik}(g) \otimes a_{kj}(g).$$

We have for all $g \in \Gamma$, $\epsilon(a_{ij}(g)) = \delta_{ij}$ and $S(a_{ij}(g)) = a_{ji}(g^{-1})$. Furthermore, $H_N^+(\Gamma)$ is isomorphic, as compact quantum group, with $\widehat{\Gamma} \wr_* S_N^+$.

With $\Gamma = \mathbb{Z}_s := \mathbb{Z}/s\mathbb{Z}$ or $\Gamma = \mathbb{Z}$, one gets the quantum reflection groups:

Definition 1.1.17. *Let $s \geq 1$ be an integer and $N \geq 2$. The quantum reflection group H_N^{s+} is the pair $(C(H_N^{s+}), \Delta)$ where $C(H_N^{s+})$ is the universal C^*-algebra generated by N^2 normal elements U_{ij} such that for all $1 \leq i, j \leq N$:*

- *(a) $U = (U_{ij})$ and ${}^tU = (U_{ji})$ are unitary,*

(b) $U_{ij}U_{ij}^*$ *is a projection,*

(c) $U_{ij}^s = U_{ij}U_{ij}^*$,

(d) $\Delta(U_{ij}) = \sum_{k=1}^N U_{ik} \otimes U_{kj}$.

- *If* $\Gamma = \mathbb{Z}$, *one gets* $H_N^{\infty+} = (C(H_N^{\infty+}), \Delta)$ *where* $C(H_N^{\infty+})$ *and* Δ *are defined as above except that one removes the relations (1c) above.*

Their fusion rules were computed by Banica and Vergnioux in [BV09]. We will use the following operations on the monoid $\langle \mathbb{Z}_s \rangle$ generated by the words over $\mathbb{Z}_s$. If $(i_1, \dots, i_k), (j_1, \dots, j_l) \in \langle \mathbb{Z}_s \rangle$,

- Concatenation : $(i_1, \dots, i_k)(j_1, \dots, j_l) = (i_1, \dots, i_k, j_1, \dots, j_l)$,
- Fusion : $(i_1, \dots, i_k) \cdot (j_1, \dots, j_l) = (i_1, \dots, i_k + j_1, \dots, j_l)$,

where the sum is taken in $\mathbb{Z}_s$ for $s \in [1, +\infty)$ (resp. $\mathbb{Z}$ if $s = \infty$).

Theorem 1.1.18. *Let* $N \geq 4$, $s \in [1, \infty]$. $C(H_N^{s+})$ *has a unique family of* N*-dimensional corepresentations (called basic corepresentations)* $\{U_k : k \in \mathbb{Z}\}$ *satisfying the following conditions:*

1. $U_k = (U_{ij}^k)$ *for any* $k > 0$.
2. $U_k = U_{k+s}$ *for any* $k \in \mathbb{Z}$.
3. $\overline{U}_k = U_{-k}$ *for any* $k \in \mathbb{Z}$.
4. U_k *is irreducible* $\forall k \neq 0$.
5. $U_0 = 1 \oplus \rho_0$, ρ_0 *irreducible.*
6. $\rho_0, U_1, \dots, U_{s-1}$ *are inequivalent corepresentations.*

Furthermore if we write for all $i \in \mathbb{Z}_s$, $\rho_i = U_i \ominus \delta_{i0}1$, *then the irreducible corepresentations of* $C(H_N^{s+})$ *can be labelled by* ρ_x *where* x *is a word in the monoid* $\langle \mathbb{Z}_s \rangle$ *and the involution and fusion rules are* $\overline{\rho}_x = \rho_{\overline{x}}$ *and*

$$\rho_x \otimes \rho_y = \sum_{x=vz,\ y=\overline{z}w} \rho_{vw} \ \oplus \sum_{\substack{x=vz,\ y=\overline{z}w \\ v\neq\emptyset, w\neq\emptyset}} \rho_{v.w}$$

The case of the fusion rules for the free wreath products $\widehat{\Gamma} \wr_* S_N^+$ for arbitrary Γ was not known except when $N = 2$:

Theorem 1.1.19. *[Bic04] Let* Γ *be a group.*

1. *To any element* $x \in \Gamma * \Gamma \setminus \{e\}$ *corresponds a two-dimensional irreducible corepresentation* v_x *of* $H_2^+(\Gamma)$. *Two such corepresentations* v_x *and* v_y *are isomorphic*

if and only if $x = y$ or $x = \tau(y)$ where τ denotes the canonical involutive group automorphism of $\Gamma * \Gamma$. There also exists a non-trivial one-dimensional irreducible corepresentation d.

2. *Any non-trivial irreducible representation of $H_2^+(\Gamma)$ is equivalent to one the irreducible corepresentations listed above.*

3. *One has the following fusion rules: $\forall x, y \in \Gamma * \Gamma \setminus \{e\}$*

$$v_x \otimes v_y = v_{xy} \oplus v_{x\tau(y)} \text{ if } x \neq y^{-1} \text{ and } x \neq \tau(y)^{-1},$$

$$v_x \otimes v_{x^{-1}} = 1 \oplus d \oplus v_{x\tau(x)^{-1}},\ d \otimes d = 1,\quad v_x \otimes d = v_x = d \otimes v_x$$

One aim of this thesis is to generalize both previous results that is to say to describe the fusion rules of the free wreath product $\widehat{\Gamma} \wr_* S_N^+$ for any (discrete group) Γ and all $N \geq 4$. The case $N = 3$ remains open.

1.1.5 C^*-tensor categories

A category $\mathscr{C}$ is a structure that includes two classes $Ob(\mathscr{C})$ and $Hom(\mathscr{C})$. The elements $a \in Ob(\mathscr{C})$ are called objects, the elements $f \in Hom(\mathscr{C})$ are called morphisms and have a unique source object $a \in Ob(\mathscr{C})$ and a unique target object $b \in Ob(\mathscr{C})$, $f : a \to b$. We denote by $Hom(a, b)$ the class of all morphisms from a to b and we set $End(a) = Hom(a, a)$.

We must add some structure on these algebraic objects to get the good framework for our purpose. We will call monoidal C^*-tensor category, a category $\mathscr{C}$ with $Ob(\mathscr{C})$ as a set of objects and such that (see e.g. [NT], Section 2):

- $\forall U, V \in Ob(\mathscr{C})$, $Hom(U, V)$ is a Banach space and $\forall U, V, W \in Ob(\mathscr{C})$, the map $(S, T) \in Hom(U, V) \times Hom(V, W) \to ST \in Hom(U, W)$ is bilinear and $||ST|| \leq ||S||\ ||T||$.
- There is an antilinear contravariant functor $* : \mathscr{C} \to \mathscr{C}$ which is the identity on $Ob(\mathscr{C})$. Contravariant here means $T \in Hom(U, V) \Rightarrow T^* \in Hom(V, U)$. We assume furthermore that $T^{**} = T$, $||T^*T|| = ||T||^2, \forall T$. In particular $End(U)$ is a C^*-algebra for all $U \in Ob(\mathscr{C})$.
- The category is monoidal that is there are a bilinear bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, a natural unitary *associativity* isomorphism $\alpha_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$, an

object $1 := 1_{\mathscr{C}}$ and natural unitary isomorphisms $\lambda_U : 1 \otimes U \to U$, $\rho_U : U \otimes 1 \to U$ such that $\forall U, V, W, X \in Ob(\mathscr{C})$:

$$(id_U \otimes \alpha_{V,W,X}) \circ \alpha_{U,V\otimes W,X} \circ (\alpha_{U,V,W} \otimes id_X) : ((U \otimes V) \otimes W) \otimes X \to U \otimes (V \otimes (W \otimes X))$$

and

$$\alpha_{U,V,W\otimes X} \circ \alpha_{U\otimes V,W,X} : ((U \otimes V) \otimes W) \otimes X \to U \otimes (V \otimes (W \otimes X))$$

are equal, and similarly,

$$id_U \otimes \lambda_V \alpha_{U\otimes 1,V} = \rho_U \otimes id_V \text{ in } Hom((U \otimes 1) \otimes V, U \otimes V).$$

- $(S \otimes T)^* = S^* \otimes T^*, \forall S, T \in Hom(\mathscr{C})$.
- $\mathscr{C}$ has finite direct sums: $\forall U, V \in Ob(\mathscr{C})$, $\exists W \in Ob(\mathscr{C})$ and isometries $u \in Hom(U, W), v \in Hom(V, W)$ such that $uu^* + vv^* = 1$.
- $\mathscr{C}$ has sub-objects: $\forall U \in Ob(\mathscr{C}), \forall p \in End(U)$ such that $p^2 = p = p^*$, $\exists V \in Ob(\mathscr{C})$ and an isometry $v \in Hom(V, U)$ such that $vv^* = p$.
- $End(1) = \mathbb{C}1$.

A C^*-tensor category C is said to be strict if for all objects $U, V, W \in Ob(\mathscr{C})$, $(U \otimes V) \otimes W = U \otimes (V \otimes W)$, $1 \otimes U = U = U \otimes 1$ i.e. all morphisms $\alpha_{U,V,W}$, λ_U, ρ_U are the identity morphisms. It is proved in [ML98] that any C^*-tensor category as above can be strictified and computations can be done as if the category were actually strict.

Let us denote as in [NT], $Hilb_f$ the C^*-tensor category of all finite-dimensional Hilbert spaces: one must restrict to a set of finite-dimensional Hilbert spaces, instead of the class of all finite-dimensional Hilbert spaces, for the category to be small (this will be the case when we deal with the category a finite dimensional corepresentations of a compact quantum group $\mathbb{G}$). The one-dimensional Hilbert space is chosen to be $\mathbb{C}$. The morphisms are linear maps between these Hilbert spaces and the discussion on the strictified categories above, allows us to work in this category as if it were a strict one, the natural associativity morphisms being the followings $(\xi \otimes \eta) \otimes \zeta \to \xi \otimes (\eta \otimes \zeta)$.

Another fundamental example that we will consider and investigate is the category of corepresentations of a compact quantum group, see Section 1.1, where the functor is given by the tensor product of corepresentations. Unlike the case of Hilbert spaces, $u \otimes v$ is not isomorphic to $v \otimes u$ in general.

A tensor functor between two C^*-tensor categories $\mathscr{C}$ and $\mathscr{C}'$, is a functor $F : \mathscr{C} \to \mathscr{C}'$, linear on morphisms and given with an isomorphism $F_0 : 1_{\mathscr{C}'} \to F(1_{\mathscr{C}})$ and a natural isomorphism $F_2 : F(U) \otimes F(V) \to F(U \otimes V)$ such that for all $U, V, W \in Ob(\mathscr{C})$, we have the following equality in the $Hom_{\mathscr{C}'}$ set

$$(F(U) \otimes F(V)) \otimes F(W) \to F(U \otimes (V \otimes W)) :$$

$$F(\alpha_{U \otimes V, W}) \circ F_2 \otimes (F_2 \otimes id_{F(W)}) = F_2 \circ (id_{F(U)} \otimes F_2) \circ \alpha_{F(U) \otimes F(V), F(W)}$$

and the following equalities in $Hom_{\mathscr{C}'}(1_{\mathscr{C}'} \otimes F(U), F(U))$ and $Hom_{\mathscr{C}'}(F(U) \otimes 1_{\mathscr{C}'}, F(U))$:

$$F(\lambda_U) \circ F_2 \circ (F_0 \otimes id_{F(U)}) = \lambda'_{F(U)},$$
$$F(\rho_U) \circ F_2 \circ (id_{F(U)} \otimes F_0) = \rho'_{F(U)}.$$

A tensor functor is said to be unitary if in addition, we have : $F(T^*) = F(T)^*$ for all morphisms T and if F_2 and F_0 are unitary.

We will say that two C^*-tensor categories $\mathscr{C}$ and $\mathscr{C}'$ are unitarily monoidally equivalent if there exist unitary tensor functors $F : \mathscr{C} \to \mathscr{C}'$ and $G : \mathscr{C}' \to \mathscr{C}$ such that FG and GF are naturally and unitarily monoidally isomorphic to the identity functor.

We will state Woronowicz's Tannaka-Krein duality theorem in Subsection 1.1.6 and we must therefore require one more assumption on the categories we consider; they must be rigid i.e. any object has a conjugate: $\bar{U}$ is said to be a conjugate to $U \in Ob(\mathscr{C})$, if there exist morphisms

$$R : 1 \to \bar{U} \otimes U \text{ and } \bar{R} : 1 \to U \otimes \bar{U}$$

such that

$$U \xrightarrow{id_U \otimes R} U \otimes \bar{U} \otimes U \xrightarrow{\bar{R}^* \otimes id_U} U \text{ and } \bar{U} \xrightarrow{id_{\bar{U}} \otimes \bar{R}} \bar{U} \otimes U \otimes \bar{U} \xrightarrow{R^* \otimes id_{\bar{U}}} \bar{U} \tag{1.3}$$

are the identity in $Hom(U, U)$ and $Hom(\bar{U}, \bar{U})$. The maps $R, \bar{R}$ are called the duality maps. We now cite the following theorem called Frobenius reciprocity theorem since we will use it several times in the sequel (see Chapter 3).

Theorem 1.1.20. *Let $\mathscr{C}$ be a monoidal rigid C^*-tensor category and let $U, \bar{U}, R, \bar{R}$ be an object, its conjugates and the morphisms satisfying the conjugate equations (1.3). Then for all $V, W \in Ob(\mathscr{C})$ the map*

$$Hom(U \otimes V, W) \to Hom(V, \bar{U} \otimes W),\ T \mapsto (id_{\bar{U}} \otimes T) \circ (R \otimes id_V)$$

is a linear isomorphism with inverse

$$S \mapsto (\bar{R}^* \otimes id_W) \circ (id_U \otimes S).$$

Similarly, $Hom(V \otimes U, W) \simeq Hom(V, W \otimes \bar{U})$.

Notice that the previous theorem holds even if we do not assume the category to be rigid but if we only assume that U has a conjugate. A corollary is that for any object U with a conjugate, $End(U)$ is finite-dimensional. In particular in a rigid monoidal category any object has a finite-dimensional End space. Notice that such categories are semi-simple: any object decomposes as a direct sum of simple objects U_α (i.e. $End(U_\alpha) = \mathbb{C}id$).

One can see that the category of all finite-dimensional Hilbert spaces is rigid considering the duality maps

$$r : \mathbb{C} \to \bar{H} \otimes H, 1 \mapsto \sum_{i \in I} \bar{e}_i \otimes e_i$$

and

$$\bar{r} : \mathbb{C} \to H \otimes \bar{H}, 1 \mapsto \sum_{i \in I} e_i \otimes \bar{e}_i,$$

where $H = \langle e_i : i \in I \rangle$ is any finite dimensional Hilbert space.

The category of unitary corepresentations of a compact quantum group is also rigid if one considers the conjugation of unitary corepresentations recalled in Subsection 1.1.1.

To conclude this section, we recall the notion of fiber tensor functor $F : \mathscr{C} \to Hilb_f$: they are tensor functors $F : \mathscr{C} \to Hilb_f$ which are injective on morphisms.

1.1.6 Woronowicz's Tannaka-Krein duality

We now combine the notions introduced in Subsection 1.1.1 and in Subsection 1.1.5.

For a given compact quantum group $\mathbb{G}$, we denote by $Rep(\mathbb{G})$, the category with

- objects: all finite dimensional unitary corepresentations U^α of $\mathbb{G}$ acting on Hilbert space H_α,
- morphisms: all intertwiners $T : H_\alpha \to H_\beta$ between corepresentations $U^\alpha, U^\beta \in Rep(\mathbb{G})$.

One can prove that it is a monoidal rigid C^*-tensor category with the obvious bifunctor, conjugates, unit object, sub-objects and direct sums.

With the previous notations, notice that there is a canonical fiber functor

$$Rep(\mathbb{G}) \to Hilb_f,$$

given by $U \mapsto H_U$ where U is the representation Hilbert space of any finite dimensional $\mathbb{G}$-corepresentation U. We can now give Woronowicz's Tannaka-Krein duality theorem:

Theorem 1.1.21. *Let $\mathscr{C}$ be a rigid monoidal C^*-tensor category, $F : \mathscr{C} \to Hilb_f$ a unitary fiber functor. Then there exist a compact quantum group $\mathbb{G}$ and a unitary monoidal equivalence $E : \mathscr{C} \to Rep(\mathbb{G})$ such that F is unitarily monoidally isomorphic to the composition of the canonical fiber functor $Rep(\mathbb{G}) \to Hilb_f$ with E.*

We say that two compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ are monoidally equivalent if $Rep(\mathbb{G}_1)$ and $Rep(\mathbb{G}_2)$ are unitarily monoidally equivalent. Monoidal equivalence transposes many interesting properties, see [DR07]. This allows for instance the authors of [VV07], [Fre13], [DCFY13] to translate algebraic and analytical properties of certain compact quantum groups to other ones.

We recall a few notions on non-crossing partitions and the linear maps which naturally arise from them.

Definition 1.1.22. *We denote by $NC(k,l)$ the set of non-crossing diagrams between k upper points and l lower points, that is the non-crossing partitions of the sets with $k+l$ ordered elements, with the following pictorial representation:*

$$\left\{ \begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \mathscr{P} & & \\ \cdot & \cdot & \cdot & \cdot & \end{array} \right\}$$

with k upper points, l lower points and $\mathscr{P}$ is a a diagram composed of strings which connect certain upper and/or lower points and which do not cross one another.

Such non-crossing partitions give rise to new ones by tensor product, composition and involution:

Definition 1.1.23. *Let $p \in NC(k,l)$ and $q \in NC(l,m)$. Then, the tensor product, composition and involution of the partitions p,q are obtained by horizontal concatenation, vertical concatenation and upside-down turning:*

$$p \otimes q = \{\mathscr{P}\mathscr{Q}\}\,,\ pq = \left\{ \begin{array}{c} \mathscr{Q} \\ \mathscr{P} \end{array} \right\} - \{\textit{closed blocks}\},\ p^* = \{\mathscr{P}^{\downarrow}\}.$$

The composition pq is only defined if the number of lower points of q is equal to the number of upper points of q. When one identifies the lower points of p with the upper *points of q, closed blocks might appear, that is strings which are connected neither to the new upper points nor to the new lower points. These blocks are discarded from the final pictorial representation. We denote by NC the collection of all non-crossing partitions.*

From non-crossing partitions $p \in NC(k,l)$ naturally arise linear maps T_p:

Definition 1.1.24. *Consider (e_i) the canonical basis of $\mathbb{C}^N$. Associated to any non-crossing partition $p \in NC(k,l)$ is the linear map $T_p \in B\left(\mathbb{C}^{N^{\otimes k}}, \mathbb{C}^{N^{\otimes l}}\right)$:*

$$T_p(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_{j_1,\dots,j_l} \delta_p(i,j) e_{j_1} \otimes \cdots \otimes e_{j_l}$$

where i (respectively j) is the k-tuple $(i_1,\dots,i_k)$ (respectively l-tuple $(j_1,\dots,j_l)$) and $\delta_p(i,j)$ is equal to:

1. 1 *if all the strings of p join equal indices,*
2. 0 *otherwise.*

Tensor products, compositions and involutions of diagrams behave as follows with respect to the associated linear maps:

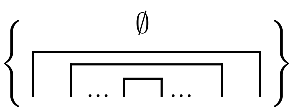
Proposition 1.1.25. *([BS09, Proposition 1.9] Let p,q be non-crossing partitions and $b(p,q)$ be the number of closed blocks when performing the vertical concatenation (when it is defined). Then:*

1. $T_{p\otimes q} = T_p \otimes T_q$,
2. $T_{pq} = n^{-b(p,q)} T_p T_q$,
3. $T_{p^*} = T_p^*$.

The Proposition 1.1.25 implies easily that the collection of spaces

$$span\{T_p : p \in NC(k,l)\}$$

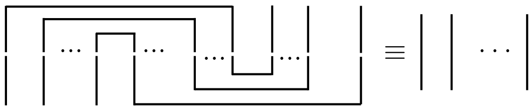
form a C^*-tensor category with $\mathbb{N}$ as a set of objects. Furthermore, this tensor category is rigid since the partitions of type

$r =$  $\in NC(\emptyset; 2k)$

are non-crossing and since the following conjugate equations hold:

$$(T_r^* \otimes id) \circ (id \otimes T_r) = id = (id \otimes T_r^*) \circ (T_r \otimes id), \tag{1.4}$$

as one can see by a direct computation or by composing the diagrams:



$$\equiv id_{(\mathbb{C}^N)^{\otimes k}}$$

and this gives the required duality map R with the notations of Section 1.1.5 (in this case $R = \bar{R}$).

We can give a result we will use many times in this thesis. We recall that we denote by v the magic unitary matrix generating $C(S_N^+)$.

Theorem 1.1.26 ([Ban99b])**.** *Let $k, l \in \mathbb{N}$. Then*

$$Hom_{S_N^+}(v^{\otimes k}; v^{\otimes l}) = span\{T_p : p \in NC(k,l)\}.$$

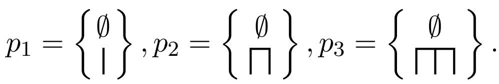
Sketch of proof. It is easy to check the inclusion $\supset$. One just has to use the properties of the magic unitary v. Conversely, applying Woronowicz's Tannaka-Krein provides a compact (matrix) quantum group $\mathbb{G} = (A, \Delta, v')$ whose Hom spaces are given by the linear maps associated to all non-crossing partitions in NC and whose underlying Woronowicz-C^*-algebra A is generated by the coefficients of a unitary matrix v'. To conclude, one has to prove that v' shares exactly the same properties as v. One can easily check, thanks to Frobenius reciprocity theorem (see Theorem 1.1.20) that v' is a magic unitary using the linear maps

$$T_{p_1} \in Hom(1, v'),$$

$$T_{p_2} \in Hom(1, (v')^{\otimes 2}) \simeq Hom\left(\overline{v'}, v'\right),$$

$$T_{p_3} \in Hom(1, (v')^{\otimes 3}) \simeq Hom\left(\overline{v'}, (v')^{\otimes 2}\right)$$

and their adjoints, where



Furthermore, it is clear that by basic operations on non-crossing partitions recalled above, one can recover any non-crossing partition in NC from p_1, p_2, p_3. Then A is not only a quotient but is actually isomorphic with $C(S_N^+)$. Then one gets that the Hom spaces between tensor products of the generating matrix of S_N^+ are described by the linear maps T_p where p runs over the collection of all non-crossing partitions in NC. □

This theorem is well known now and generalizes the analogue result in the case of $C(S_N)$, where S_N designates the classical permutation group where any (non-crossing or crossing) partition gives rise to an intertwiner. We refer the reader to the remark in [BV09] after Theorem 5.5 and e.g. [Spe97] for more informations on non-crossing partitions and this passage from classical to free probabilities.

In the next chapters, we will have to consider certain non-crossing partitions with additional properties. For instance, one can describe the intertwiner spaces in H_N^{s+} as follows (see Section 1.1.4 for the definition of quantum reflection groups):

Theorem 1.1.27 ([BV09]). *Let $i_1, \dots, i_k, j_1, \dots, j_l \in \mathbb{Z}_s$ then:*

$$Hom_{H_N^{s+}}(U_{i_1} \otimes \cdots \otimes U_{i_k}, U_{j_1} \otimes \cdots \otimes U_{j_l}) = span\{T_p : p \in NC_s(\underline{i}, \underline{j})\}$$

where p is an element of $NC_s(\underline{i}, \underline{j}))$ if and only if it is a non-crossing partition in $NC(k, l)$ satisfying the additional rule: if one puts the k-tuple $\underline{i}$ on the k-upper points of p and $\underline{j}$ on the l-lower points then, in each block, the sum on the i-indices must be equal to the sum on the j-indices modulo s (if $s = \infty$ we take tuples in $\mathbb{Z}$ and make the convention that equalities modulo s are equalities).

Sketch of proof. Similar arguments as above for NC show that the collection of spaces $span\{T_p : p \in NC_s(\underline{i}, \underline{j})\}$ form a rigid C^*-tensor category. One can prove, applying Woronowicz's Tannaka-Krein duality, that one recovers the quantum reflection groups H_N^{s+} (see [BV09]). More precisely, since $C(H_N^{s+})$ is generated by the coefficients of tensor products of the generating matrix U and its conjugate $\bar{U}$, one can first focus on the Hom spaces $Hom_{H_N^{s+}}(U^{\epsilon_1} \otimes \cdots \otimes U^{\epsilon_k}, U^{\eta_1} \otimes \cdots \otimes U^{\eta_l})$ with $\epsilon \in \{1, -\}$. This is done in [BBCC11], where the authors deduce these intertwiner spaces from the ones in (A_k, Δ) (computed in [Ban08] and [BBC07]). Recall that A_k is the universal C^*-algebra generated by N^2 elements W_{ij} such that W and $\bar{W}$ are unitary $ab^* = a^*b = 0$ on for a, b any element on the same row or column of W. The intertwiner spaces in A_k between tensor products of $W, \overline{W}$ corresponds again to certain linear maps obtained from diagrams. They are colorable diagrams p such that if one puts W colored x, y and $\overline{W}$ colored y, x on the points of p then the strings of p must match the colors. It is easy to check that $C(H_N^{s+})$, $s \in [1, +\infty]$ is a quotient of the A_k and that the normality conditions satisfied by the generators U_{ij} of $H_N^{\infty+}$ are equivalent to the fact that $T_p \in Hom(W \otimes \overline{W}, \overline{W} \otimes W)$ with $p = \{H\}$. Then the category of morphisms in H_N^{s+} is generated by the one in A_k and by the morphisms induced by p. This allows to deduce the theorem. $\square$

We will use similar arguments to deduce the intertwiner spaces in $H_N^+(\Gamma)$ from the ones of certain free product quantum groups $(H_N^{\infty+})^{*p}$, see Chapter 3.

1.2 Operator algebraic properties for quantum groups

1.2.1 Group algebras and approximation properties

Many results and clear expositions of the following notions are contained in [BO08] and [CCJ+].

Amenability To any discrete group Γ, one can associate a full (also named universal) group C^*-algebra as follows:

$$C^*(\Gamma) = \overline{\mathbb{C}[\Gamma]}^{||\cdot||_u}$$

where $||\cdot||_u$ is the universal C^*-norm defined by:

$$||x||_u = \sup_\pi \{||\pi(x)|| \text{ such that } \pi : \mathbb{C}[\Gamma] \to B(H_\pi) \text{ is a } *\text{-representation}\}.$$

One can also complete the group $*$-algebra $\mathbb{C}[\Gamma]$ with respect to the norm implemented by the left regular representation

$$\lambda : \Gamma \to B(l^2(\Gamma)), \lambda(s)\delta_t = \delta_{st}.$$

One puts:

$$C_r(\Gamma) = \overline{\mathbb{C}[\Gamma]}^{||\cdot||_r}$$

where

$$||x||_r = ||\lambda(x)||_{B(l^2(\Gamma))}.$$

The universal C^*-algebra $C^*(\Gamma)$ satisfies a universal property and in particular there is a canonical surjective $*$-homomorphism $C^*(\Gamma) \to C_r(\Gamma)$ given by λ. One says that Γ is amenable if λ is injective, that is $C^*(\Gamma) = C_r^*(\Gamma)$.

Amenability admits many other reformulations including the following ones:

- the trivial representation is weakly contained in the left regular representation λ: $\exists(\xi_i)_i \in l^2(\Gamma)$, $||\xi_i||_2 = 1$, such that $||\lambda_s(\xi_i) - \xi_i||_2 \to 0, \forall s \in \Gamma$;

- there exists a net $(\phi_i)_i$ of finitely supported positive definite functions ($\forall g_k \in \Gamma$, $(\phi(g_k^{-1}g_l))_{k,l}$ is positive) on Γ such that $\phi_i \to 1$ pointwise:

- there exists a left invariant mean: $\exists$ a state $m : l^\infty(\Gamma) \to \mathbb{C}$ such that $m(fg) = m(g), \forall f, g \in l^\infty(\Gamma)$.

- $C_r^*(\Gamma)$ (or equivalently $C^*(\Gamma)$) is nuclear: there exist nets of completely positive maps $\alpha_i : C_r^*(\Gamma) \to M_{k(i)}(\mathbb{C})$, $\beta_i : M_{k(i)} \to C_r^*(\Gamma)$ such that $\beta_i \circ \alpha_i(x) \to_i x, \forall x$.

Compact and abelian groups are basic examples of amenable groups. However, this property fails for free groups.

Haagerup approximation property The non abelian free group on two generators $F_2 = \langle a, b\rangle$ is not amenable. One can see it thanks to the invariant mean characterization recalled above and considering the subsets $A^\pm, B^\pm \subset F_2$ of the reduced words starting by a, a^{-1}, b, b^{-1}. However the free groups $F_n, n \geq 1$ possess another approximation property known as the Haagerup approximation property. As for the amenability, this approximation property has many equivalent formulations.

Let us first recall a few notions.

Definition 1.2.1. *Let Γ be a discrete group and let (π, H) be a unitary representation of Γ. We say that a function $b : \Gamma \to H$ is a 1-cocycle on Γ if for all $s, t \in \Gamma$:*

$$b(st) = b(s) + \pi(s)b(t).$$

We say that b is metrically proper if the map $\Gamma \to \mathbb{R}, s \mapsto ||b(s)||$ is proper, that is for all $K > 0$,

$$\#\{s \in \Gamma : ||b(s)|| < K\} < \infty.$$

Proposition 1.2.2. *A function $b : \Gamma \to H$ is a 1-cocycle if and only if the map $\theta : \Gamma \to Aff(H)$ defined for all $s \in \Gamma$ and all $\xi \in H$ by*

$$\theta(s)(\xi) = \pi(s)\xi + b(s)$$

is a group homomorphism. $Aff(H)$ designates the group of affine isometries on H. Any group homomorphism from Γ into $Aff(H)$ for some Hilbert space is of this form for a certain representation $\pi : \Gamma \to U(H)$.

We have recalled the definition of positive definite functions above. There is a (conditionally) negative definite counterpart to this definition:

Definition 1.2.3. *A conditionally negative definite function on a discrete group Γ is a function $\psi : \Gamma \to \mathbb{C}$ such that for all n, all $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ with $\sum_{i=1}^{n} \lambda_i = 0$, for all $s_1, \ldots, s_n \in \Gamma$, one has*

$$\sum_{i,j} \bar{\lambda}_i \lambda_j \psi(s_i^{-1} s_j) \leq 0.$$

We say that ψ is proper if $\psi(s) \to \infty$ when $s \to \infty$.

Both notions of positive and conditionally negative definite functions are linked by the following theorem.

Theorem 1.2.4. *(Schoenberg's Theorem) Let $\psi : \Gamma \to \mathbb{R}$ be a function such that $\psi(e) = 0$ and $\psi(s) = \psi(s^{-1})$ for all $s \in \Gamma$. Then the following assertion are equivalent:*

- *ψ is a conditionally negative definite function,*
- *the function $exp(-t\psi)$ is positive definite for all $t \geq 0$.*

Theorem 1.2.5. *Let Γ be a discrete group. The following assertions are equivalent and if Γ satisfies one of the following condition, we say that Γ has the Haagerup property:*

- *There exists a unitary c_0-representation of Γ, $\pi : \Gamma \to U(H)$ $(\langle \pi(s)\xi, \eta \rangle \to 0, \forall \xi, \eta \in H)$ weakly containing the trivial.*
- *There exists a net (ϕ_i) of positive definite functions on Γ such that $\phi_i(e) = 1$, ϕ_i vanishes at infinity and $\phi_i \to 1$ pointwise.*
- *There exists a proper conditionally negative definite function $\psi : G \to \mathbb{R}_+$.*
- *Γ admits a metrically proper 1-cocycle.*
- *Γ admits a metrically proper affine isometric action on a real Hilbert space.*

Recall that c_0-representation are also called mixing. Obviously, amenable groups have the Haagerup property. A well known result is the fact that a group acting faithfully on a tree has the Haagerup property. Indeed, the action of Γ on the l^2-space of the edges admits a proper 1-cocycle $b : \Gamma \to l^2(E)$. The free groups F_N have the Haagerup property as one can see thank to their action on their Cayley tree.

The class of groups having the Haagerup property is closed under taking subgroups, free products, increasing unions. This property is also closed under taking wreath products (see e.g. [CSV12]): $H \wr G = H^{(G)} \ltimes G$ where

$$H^{(G)} := \{\text{functions } f : G \to H : |supp(f)| < \infty\}.$$

This is proved in [CSV12] using the notion of wall space structures. It is one purpose of this thesis to prove, in certain cases, a quantum analogue of this result.

We will denote by $L(\Gamma)$ the von Neumann algebra generated by $C_r(\Gamma)$: $L(\Gamma) = C_r(\Gamma)''$. It is a well known result that Γ has the Haagerup property if and only if $L(\Gamma)$ has the Haagerup property (see Subsection 1.2.3 for the definition of this property in the finite von Neumann setting). In this thesis, we will study the Haagerup property for certain von Neumann algebras associated to compact quantum groups of Kac type. This is consistent in view of Theorem 1.2.12.

1.2.2 Amenability for quantum groups

In this subsection, we give an overview of the results that one can find in particular in [BMT01] and [Tom06]. Recall first that there is a general notion of operator amenability for Banach algebras. The concept of amenability for Kac algebras was then introduced by Voiculescu [Voi79] via the existence of a left invariant mean. This was a priori a weaker form of amenability. In the setting of Hopf von Neumann algebras, Ruan proved that both notions of amenability coincide and are also equivalent to the so called strong Voiculescu amenability in the case of discrete Kac algebras [Rua96].

Recall that if $G = (C(\mathbb{G}), \Delta)$ is a compact quantum group, the counit $\epsilon : Pol(\mathbb{G}) \to \mathbb{C}$ satisfies $\epsilon(u_{ij}) = \delta_{ij}$ for any coefficient u_{ij} of a corepresentation $u \in Rep(\mathbb{G})$. We denote by $C_r(\mathbb{G})$ and $C_u(\mathbb{G})$ the reduced and universal version of the C^*-algebra $C(\mathbb{G})$. Because of the definition of $C_u(\mathbb{G})$, the counit is bounded on $C_u(\mathbb{G})$ but this may be not the case on $C_r(\mathbb{G})$.

Definition 1.2.6. *A compact quantum group $\mathbb{G} = (C(G), \Delta)$ is said to be coamenable, if the counit is bounded on the reduced C^*-algebra $C_r(G)$.*

This definition is motivated by compact quantum groups arising from the classical discrete groups case $(C^*(\Gamma), \Delta)$, $\Delta(g) = g \otimes g, \forall g \in \Gamma$. In Subsection 1.2.1, we have seen above that Γ is amenable if and only if the trivial representation weakly contains the regular one. This is equivalent to the fact that the counit $\epsilon(g) = \delta_{g,e}$ is norm-bounded on $C_r^*(\Gamma)$. We have recalled in Subsection 1.2.1 that the free group $\Gamma = F_2$ is not amenable because one can not find an invariant mean on $l^\infty(F_2)$. One can also use the fact that $C_r^*(F_2)$ is simple [Pow75], which implies that any $*$-homomorphism $\rho : C_r^*(F_2) \to \mathbb{C}$ vanishes. As a consequence, the counit $\epsilon : \Gamma \to \mathbb{C}$ can not be norm-bounded on $C_r^*(F_2)$.

On the other hand, the examples arising from classical compact groups $(C(G), \Delta)$

$$\Delta(f)(x, y) = f(xy), \forall x, y \in G$$

are coamenable since in this case, the counit is given by the map $f \mapsto f(e)$ and the Haar state is faithful.

The previous definition of coamenability is equivalent (see [BMT01]) to the fact that, as in the classical case, the canonical surjective morphism $C_u(\mathbb{G}) \to C_r(\mathbb{G})$ is an isomorphism. More precisely, if $\mathbb{G} = (C(\mathbb{G}), \Delta)$ is a compact quantum groups with Haar state h and counit $\epsilon : Pol(\mathbb{G}) \to \mathbb{C}$, then the following are equivalent:

- $(C(\mathbb{G}), \Delta)$ is coamenable (ϵ is bounded on $C_r(\mathbb{G})$).
- the canonical map from $C_u(\mathbb{G})$ to $C_r(\mathbb{G})$ is a $*$-isomorphism.
- The Haar state h is faithful on $C_u(\mathbb{G})$.

We recall the definition of the twisted $SU_q(2)$ compact quantum group (see [Wor87b]). They are the compact matrix quantum groups generated by the entries of the fundamental unitary corepresentation $\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$. This matrix is unitary if and only if

$$\alpha^*\alpha + \gamma^*\gamma = 1,\ \gamma^*\gamma = \gamma\gamma^*$$
$$\alpha^*\alpha + q^2\gamma^*\gamma = 1,\ \alpha\gamma = q\gamma\alpha$$
$$\alpha\gamma^* = q\gamma^*\alpha.$$

The q-deformations $SU_q(2)$ are coamenable for all $q \neq 0$ and of non-Kac type for $q \neq \pm 1$. A crucial tool for the proof of this fact is the theory of corepresentations of these compact quantum groups that one can find in [Wor98].

It is more consistent to call a quantum group *amenable* when one deals with a discrete (or locally compact) quantum group to stick with the classical situation. This is why the vocable *coamenable* was introduced in the compact setting. The common definition for Kac algebras we mentioned above can be extended to locally compact quantum groups and it is equivalent, in the unimodular and discrete case, to the above definition via the duality we explained before.

Definition 1.2.7. *A discrete quantum group* $\widehat{\mathbb{G}} = (l^\infty(\widehat{\mathbb{G}}), \widehat{\Delta})$ *is said to be amenable, if it admits an invariant mean that is a state* m *on* $l^\infty(\widehat{\mathbb{G}})$ *such that*

$$m((\omega \otimes id)\Delta(x)) = \omega(1)m(x),$$

$$m((id \otimes \omega)\Delta(x)) = \omega(1)m(x),$$

for all $x \in c_0(\widehat{\mathbb{G}})$ *and all* $\omega \in l^\infty(\widehat{\mathbb{G}})_*$.

One can find in [Tom06], a summary of the previous work on (co)amenability for quantum groups in [BMT01], [BMT02], [BMT03], [BCT05] and the striking result with the invariant mean formulation without the Kac type assumption:

Theorem 1.2.8. *([Tom06]) Let* $\widehat{\mathbb{G}} = (c_0(\widehat{\mathbb{G}}), \widehat{\Delta})$ *be a discrete quantum group. Then the following are equivalent:*

1. $\widehat{\mathbb{G}}$ *is amenable.*
2. $\mathbb{G}$ *is coamenable.*

In the unimodular case, the assertions above hold if and only if $C(\mathbb{G})$ is nuclear and also if and only if $L^\infty(\mathbb{G})$ is injective.

However, in the non-Kac/non-unimodular case, it is still open whether nuclearity of $C(\mathbb{G})$ or injectivity of $L^\infty(\mathbb{G})$ imply the amenability of $\widehat{\mathbb{G}}$. We will come back to the definition of injectivity and the links with structure results for von Neumann algebras associated with free compact quantum groups in Section 1.3.

The basic examples of free compact quantum groups of Kac type we recalled in Subsection 1.1.4 are in most cases non coamenable: U_N^+ is not coamenable for all $N \geq 2$, O_N^+ is not coamenable for all $N \geq 3$, S_N^+ is not coamenable for all $N \geq 5$ (see [Ban97], [Ban99b]). The proofs of these facts use a "Kesten"-criterion for compact matrix quantum groups [Ban99a]. Notice that this criterion was extended to all compact quantum groups [Kye08].

As in the classical case, amenability seems too strong to include several basic examples. The Haagerup property for the above examples of compact quantum groups was proved recently by Brannan.

1.2.3 Haagerup approximation property

For a long time, the Haagerup property for free quantum groups was not known. Brannan first proved in [Bra12a] the Haagerup property for $L^\infty(O_N^+)$ and $L^\infty(U_N^+)$ and later on for $L^\infty(S_N^+)$. However, they were the only non-amenable quantum groups with this property. It is an aim of this thesis to prove the Haagerup approximation property for other examples of free compact quantum groups.

We first recall the definition of the Haagerup property for finite von Neumann algebras equipped with a faithful normal trace, see [Cho83]:

Definition 1.2.9. *Let (M, τ) be a finite von Neumann algebra with separable predual and faithful normal trace τ. M has the Haagerup property with respect to τ if there exists a net $(\phi_x)_{x\in\Lambda}$ of normal, unital and completely positive (NUCP) maps $\phi_x : M \to M$ such that*

1. *$\tau \circ \phi_x = \tau$ and the L^2-extension $\phi_x \in B(L^2(M))$ is compact,*
2. *$||\phi_x(a) - a||_2 \to 0$ for all $a \in M \hookrightarrow L^2(M)$.*

One can relax the condition $\tau \circ \phi_x = \tau$ to $\tau \circ \phi_x \leq \tau$ and the definition does not depend on the choice of the faithful trace M.

Given a compact quantum group $\mathbb{G}$ of Kac type, Brannan introduced a method to construct nets of NUCP and trace preserving maps on $L^\infty(\mathbb{G})$ starting from states on the central algebra of $\mathbb{G}$, that is the unital C^*-algebra generated by the characters of the irreducible $\mathbb{G}$-corepresentations. We denote by $L^2_\alpha(\mathbb{G})$ the image in the GNS representation of the space of coefficients of the irreducible corepresentation $\alpha \in Irr(\mathbb{G})$ of dimension d_α and by p_α the orthogonal projection of $L^2(\mathbb{G})$ onto $L^2_\alpha(\mathbb{G})$. We still denote by h, Δ the Haar state and coproduct on the reduced C^*-algebra $C_r(\mathbb{G})$ and their extension to the von Neumann algebra $L^\infty(\mathbb{G})$.

Theorem 1.2.10. *([Bra12a]) Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group of Kac type and consider the unital C^*-subalgebra $C(\mathbb{G})_0 = C^* - \langle \chi_\alpha : \alpha \in Irr(\mathbb{G})\rangle$ generated by the irreducible characters $\chi_\alpha = \sum_{i=1}^{d_\alpha} \alpha_{ii}$. Then, for any state $\psi : C(\mathbb{G}) \to \mathbb{C}$ the map*

$$T_\psi = \sum_{\alpha\in Irr(\mathbb{G})} \frac{\psi(\chi_{\bar{\alpha}})}{d_\alpha} p_\alpha$$

is a unital contraction on $L^2(\mathbb{G})$ and the restriction $T_\psi|_{L^\infty(\mathbb{G})}$ is a NUCP h-preserving map.

The proof of this result relies on an averaging of the convolution operator $(id \otimes \psi) \circ \Delta : C(\mathbb{G}) \to C(\mathbb{G})$ which factorizes to a map $C_\psi : C_r(\mathbb{G}) \to C_r(\mathbb{G})$. A calculation on the coefficients of the irreducible corepresentations allows to see that

$$\Delta^{-1} \circ E \circ ((\kappa \circ C_\psi \circ \kappa) \otimes id) \circ \Delta = T_\psi$$

where $E : L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{G}) \to \Delta(L^\infty(\mathbb{G}))$ is the normal conditional expectation which satisfies for all corepresentations $\alpha, \beta \in Irr(\mathbb{G})$ (see also [CFK12]),

$$E(\alpha_{ij} \otimes \beta_{kl}) = \frac{\delta_{\alpha,\beta}\delta_{j,k}}{d_\alpha} \Delta(\alpha_{il}).$$

This averaging method, combined with a study of the central algebra of the free orthogonal and quantum permutation group, allowed Brannan to prove that:

Theorem 1.2.11. *([Bra12a],[Bra12b])*

1. *For all $N \geq 2$ $L^\infty(O_N^+)$ has the Haagerup approximation property.*

2. *Let B be a finite dimensional C^*-algebra with δ-trace τ. Then $L^\infty(\mathbb{G}_{aut}(B,\tau))$ has the Haagerup approximation property.*

Sketch of proof. The states ψ_x that one can consider as in Theorem 1.2.10 are the evaluation states $ev_x, x \in I_N$ with $I_N = (-N, N)$ in the case of O_N^+ and $I_N = (0, N)$ in the case of S_N^+ (or more generally for all Kac type quantum automorphisms groups). Indeed, thanks to the commutative fusion rules binding the irreducible corepresentations indexed by $\mathbb{N}$ of these compact quantum groups (see Subsection 1.1.4), one can show that the central algebras $C(O_N^+)_0$ and $C(S_N^+)_0$ are commutative and isomorphic with $C([-N, N])$ and $C([0, N])$, respectively (see below). When one makes $x \to N$, these states converge to the counit, $\epsilon(\chi_\alpha) = d_\alpha$. Then, it is easy to see that the maps $(T_{\psi_x})_{x\in I_N}$ converge to the identity in L^2-norm. It remains to prove that their L^2-extension are compact operators. To do this, since the operators p_α have finite rank, one only has to prove that their eigenvalues $\dfrac{\psi_x(\chi_n)}{d_n}$ are in $c_0(\mathbb{N})$, $x \in I_N$.

Proving $\dfrac{\psi_x(\chi_n)}{d_n} \in c_0(\mathbb{N})$ then follows from certain estimates on Tchebyshev polynomials. Indeed, as mentioned above the commutativity satisfied by the fusion rules in these compact quantum group yields the following isomorphisms:

$$C(O_N^+)_0 \xrightarrow{\sim} C([0, N]), \chi_k \mapsto A_k(x)$$

$$C(S_N^+)_0 \xrightarrow{\sim} C([-N, N]), \chi_k \mapsto A_{2k}(\sqrt{x}).$$

where $(A_k)_{k\in\mathbb{N}}$ designates the family of dilated Tchebyshev polynomials, defined recursively by $A_0 = 1, A_1 = X, A_1A_k = A_{k+1} + A_{k-1}$. Under the above identifications one can prove that

$$\frac{\psi_x(\chi_n)}{d_n} = \frac{A_{pk}(x^{1/p})}{A_{pk}(N^{1/p})},$$

with $p = 1$ in the case of O_N^+ and $p = 2$ in the case of S_N^+. To conclude, one uses as announced certain estimates on these polynomials: $\dfrac{A_k(x)}{A_k(N)} \leq C_{x_0}\left(\dfrac{x}{N}\right)^k$ for some constant C_{x_0} only depending on some fixed $2 < x_0 < 3$. $\square$

Notice that the Haagerup property for the dual of U_N^+ can be obtained by using the fact that the free complexification of O_N^+ is U_N^+. We will adapt these methods and estimates in Chapter 2 to prove that the von Neumann algebras $L^\infty(H_N^{s+})$ have the Haagerup approximation property for all $s \in [1, +\infty)$ and all $N \geq 4$.

Given a state τ on a von Neumann algebra M, one can ask whether (M, τ) has the Haagerup approximation property, as in Definition 1.2.9. In the non-tracial situation, the property may rely on the chosen state τ. In the quantum setting, this implies that one can ask whether (the dual) of a non-Kac type compact quantum group has the Haagerup approximation property. This was investigated in [DFSW13], where several characterizations of the Haagerup property are proved to be equivalent for compact quantum groups. They are equivalent, in the Kac type setting, to the fact that $L^\infty(\mathbb{G})$ has the Haagerup property.

Theorem 1.2.12. *([DFSW13]) Let $\mathbb{G}$ be a compact quantum group. The following statements are equivalent and if one holds one says that $\widehat{\mathbb{G}}$ has the Haagerup approximation property.*

1. *$\mathbb{G}$ has a mixing representation weakly containing the trivial.*
2. *There exists a convolution semigroup of states $(\mu_t)_{t\geq 0}$ on $C_u(\mathbb{G})$ such that each $a_t := (\mu_t \otimes id)(W) \in c_0(\widehat{\mathbb{G}}))$ and a_t tends to 1 as $t \to 0$.*
3. *$\widehat{\mathbb{G}}$ admits a proper real cocycle.*

In the above equivalence, the unimodularity of $\widehat{\mathbb{G}}$ is not required. The above statements all imply that $L^\infty(\mathbb{G})$ has the Haagerup approximation property and the converse is true in the Kac case. In this thesis, we will deal with Kac type compact quantum groups $\mathbb{G}$. Therefore, in the statements concerning the Haagerup property, we will say that the dual of $\mathbb{G}$ has the Haagerup property and will actually prove that $L^\infty(\mathbb{G})$ has the Haagerup property.

1.2.4 Further operator algebraic properties

In this subsection, we give an overview of certain recent results concerning operator algebraic properties for quantum groups and we try to explain how one could tackle related problems with regards to the free wreath product quantum groups studied in this thesis.

In Subsection 1.2.2, we mentioned the notion of injectivity for a finite von Neumann algebra with faithful normal trace (M, τ). We may recall a few facts on the subject. Definitions and proofs can be found in [Con76] or in [Con90a].

Definition 1.2.13. *Let (M, τ) be a finite von Neumann algebra with trace τ acting on an Hilbert space H.*

- *M is said to be amenable if there exists an M-central state φ on $B(H)$ such that $\varphi|_M = \tau$.*
- *M is said to be injective if there exists a conditional expectation $E : B(H) \to M$.*

Actually a finite von Neuman (M, τ) is injective if and only if it is amenable. Furthermore, one can easily show that any hyperfinite von Neumann algebra is amenable. The converse is due to Connes.

Theorem 1.2.14. *(Connes) A finite von Neumann algebra is hyperfinite if and only if it is amenable.*

The (easy) direct implication actually provides as a nice corollary that the free group factor $L(F_N)$ is not isomorphic with the unique hyperfinite II_1-factor. Indeed, a discrete group Γ is amenable if and only if $L(\Gamma)$ is amenable. However, the original proof comes from the fact that the free group factor does not have the property Γ, i.e. that there are no bounded asymptotically central sequences which are not asymptotically trivial. Since injectivity is equivalent to hyperfiniteness and implies the property Γ, one gets the result.

We will see in next section that this property Γ is linked to recent developments on structure results for (quantum group) von Neumann algebras. Indeed, we recall here results by Isono ([Iso12], [Iso13]) and De Commer, Freslon, Yamashita ([DCFY13]). Let us begin by the definition of (weak-$*$) Completely Bounded Approximation Property for discrete quantum groups. Let $\widehat{\mathbb{G}}$ be a discrete quantum group. The Herz-Schur multiplier associated to $a \in l^\infty(\widehat{\mathbb{G}})$ is given by $(m_a \otimes id)(u_x) = (1 \otimes ap_x)u_x$ for all $x \in Irr(\mathbb{G})$ acting on H_x, where p_x is the minimal central projection in $l^\infty(\widehat{\mathbb{G}})$ corresponding to the identity on $B(H_x)$.

Definition 1.2.15. *Let $\widehat{\mathbb{G}}$ be a discrete quantum group and M be a von Neumann algebra with separable predual.*

- *We say that $\widehat{\mathbb{G}}$ is weakly amenable if there exists a net $(a_i) \subset l^\infty(\widehat{\mathbb{G}})$ such that*
  - *$a_i p_x = 0$ except for finitely many $x \in Irr(\mathbb{G})$ (finite support).*
  - *$a_i p_x$ converges to p_x in $B(H_x)$ for all $x \in Irr(\mathbb{G})$ (p.w. convergence to 1).*
  - *$\limsup_i ||m_{a_i}||_{cb}$ is finite.*

- *We say that M has the W*CBAP if there exists a net $(\psi_i)_i$ of normal, c.b. maps on M such that*

  - *$\psi_i : M \to M$ is normal and completely bounded.*
  - *ψ_i has finite rank and (ψ_i) converges p.w. to id in the σ-weak topology.*
  - $\limsup ||\psi_i||_{cb} < \infty$.

  If the above conditions are satisfied, we put (and call them Cowling-Haagerup constant)

$$\Lambda_{c.b}(\mathbb{G}) = \inf\{\limsup ||m_{a_i}||_{cb}, (a_i) \text{ as above}\},$$

$$\Lambda_{c.b}(M) = \inf\{\limsup ||\psi_i||_{cb}, (\psi_i) \text{ as above}\}.$$

Theorem 1.2.16. *([FW13],[DCFY13]) The following hold.*

1. *$L^\infty(U_F^+)$ has the W*CBAP with Cowling-Haagerup constant 1 for any matrix $F \in GL_N(\mathbb{C})$.*
2. *$L^\infty(O_F^+)$ has the W*CBAP with Cowling-Haagerup constant 1 for any matrix $F \in M_N(\mathbb{C})$ such that $F\bar{F} \in \mathbb{R}I_N$.*
3. *$L^\infty(\mathbb{G}_{aut}(B,\tau))$ has the W*CBAP with Cowling-Haagerup constant 1 for any δ-form ψ on a finite-dimensional C^*-algebra B.*

To prove this result, the authors use the monoidal equivalence between O_F^+ and $SU_q(2)$ for a certain q satisfying $q + q^{-1} = \mp Tr(F^*F)$ if $F\bar{F} = \pm I_N$. Then, they prove a central approximation property for the $SU_q(2)$ quantum groups. The property also holds for the discrete quantum groups in Theorem 1.2.16 above by monoidal equivalence, complexification and stability of this property by subgroups and free products. As a corollary, one has:

Corollary 1.2.17. *([Iso13]) Any of the following von Neumann algebras has no Cartan subalgebras if it is non-injective.*

1. *$L^\infty(U_F^+)$ for any invertible $F \in GL_N(\mathbb{C})$,*
2. *$L^\infty(O_F^+)$ for any $F \in M_N(\mathbb{C})$ such that $F\bar{F} \in \mathbb{R}I_N$,*
3. *$L^\infty(\mathbb{G}_{aut}(B,\tau))$ for a δ-form ψ on a finite-dimensional C^*-algebra B.*

Notice that the hypothesis on the injectivity for $L^\infty(U_F^+)$ is redundant since Vaes proved that it is full via the $14-\epsilon$ method, see an Appendix in [DCFY13]. We will adapt this method in Chapter 3 to obtain the fullness of the von Neumann algebras $L^\infty(H_N^+(\Gamma))$ for any discrete group Γ.

1.3 Problems and projects

1.3.1 Fusion rules for free wreath products

In this thesis, we describe the fusion rules of the free wreath products $\widehat{\Gamma} \wr_* S_N^+$ for any discrete group Γ and $N \geq 4$, Theorem 3.2.25. I am currently investigating the free wreath products $\mathbb{G} \wr_* S_N^+$ for any compact matrix quantum group $\mathbb{G}$ and $N \geq 4$. We can tackle this problem by first describing the intertwiner spaces between certain basic corepresentations indexed by the equivalence classes of $\mathbb{G}$-representations. The underlying objects are again certain non-crossing partitions $p \in NC$. Instead of being decorated by the elements of the group Γ as in the case $\widehat{\Gamma} \wr_* S_N^+$, they are non-crossing partitions where the points are decorated by $\mathbb{G}$-representations and the blocks are decorated by $\mathbb{G}$-morphisms. Then one can adapt the method of this thesis to obtain the fusion rules of such free wreath products. One could then deduce informations on the moments of characters of free wreath products $\mathbb{G} \wr_* S_N^+$ in view of Conjecture 3.1 in [BB07].

Another project that one could investigate would be to try and describe the fusion rules of $\widehat{\Gamma} \wr_* \mathbb{G}_{aut}(B, \tau)$ for any discrete group Γ and any finite dimensional C^*-algebra B equipped with a δ-form. This could provide new non-Kac examples of free wreath products.

1.3.2 Operator algebraic properties for free wreath products

We prove in this thesis that the von Neumann $L^\infty(\widehat{\Gamma} \wr_* S_N^+)$ is full and thus that it is non-injective for any discrete group Γ and $N \geq 8$. Actually, non-injectivity can be easily deduced from the fact that the compact quantum group $\widehat{\Gamma} \wr_* S_N^+$ is not coamenable because S_N^+ is not coamenable, for all $N \geq 5$. However, as mentioned above, one could consider the free wreath product quantum groups $\widehat{\Gamma} \wr_* \mathbb{G}_{aut}(B, \tau)$. If τ is not a trace, the compact quantum group $\mathbb{G}_{aut}(B, \tau)$ is not of Kac type and the non-coamenability is not enough to conclude to the non-injectivity of the associated von Neumann algebra. However, non-injectivity could be obtained in the same way we proceed in this thesis (adapted $14 - \epsilon$ methods) provided that $L^\infty(\mathbb{G}_{aut}(B, \tau))$ has not the property Γ which is only known in the tracial case, [Bra12b]. However, in this paper the author does not use the tracial property of the Haar state to prove this property.

To conclude, notice that for free wreath product quantum groups $H_N^+(\Gamma) = \widehat{\Gamma} \wr_* S_N^+$ (and more generally the free wreath products by any quantum automorphism group), natural candidates for central multiplier operators on $Pol(H_N^+(\Gamma))$ are given by

$$\sum_{\alpha\in Irr(H_N^+(\Gamma))}\prod_{i=1}^{k_\alpha}\frac{A_{l_i}\sqrt{q^w+q^{-w}}}{A_{l_i}(\sqrt{N})}p_\alpha, \qquad \text{if one writes } \alpha\in Irr(H_N^+(\Gamma)),\ \alpha=a^{l_1}z_{g_1}\dots a^{l_{k_\alpha}},$$

with $q+q^{-1}=N$ and the notation of Chapter 2 and [DCFY13].

The problem is then two fold:

1. Proving that these central multipliers yield the central approximation property as in [DCFY13] when Γ is finite.
2. Improving these central multipliers to prove the central approximation property for infinite groups.

The second question is of course related to a question arising from some results of the next two chapters. Indeed we prove that $L^\infty(H_N^+(\Gamma))$ has the Haagerup approximation property when Γ is finite but the methods presented in this thesis fail when Γ is infinite essentially because the function $L(\alpha)=\sum_i l_i, \forall\alpha=a^{l_1}z_{g_1}\dots a^{l_{k_\alpha}}$ is not proper when Γ is infinite. The archetypal example one could consider is the quantum reflection group $H_N^{\infty+}=\widehat{\mathbb{Z}}\wr_* S_N^+$.

The study of this property together with bi-exactness properties and combined with the results of Isono [Iso13], would allow in particular to conclude to strong solidity and thus to the absence of Cartan subalgebras.

Another strategy to study the operator algebraic properties of the free wreath product quantum groups $\widehat{\Gamma}\wr_* S_N^+$, would be to try and prove monoidal equivalence with a more tractable compact quantum group and then use similar techniques as in [DCFY13]. If $N=n^2$ is a square, one could try and prove that $\widehat{\Gamma}\wr_* S_{n^2}^+$ is monoidally equivalent to $\mathbb{H}=(C(\mathbb{H}),\Delta)$, where

$$C(\mathbb{H}):=C^*-\langle V_{ij}gV_{kl}: g\in\Gamma, 1\le i,j,k,l\le n\rangle\subset C^*(\Gamma)*C(O_n^+),$$

and V is the generating representation of O_n^+. Notice that $C(\mathbb{H})$ is generated by the $\widehat{\Gamma}*O_N^+$-representations, $V\otimes g\otimes V, g\in\Gamma$. This project is motivated by the fact that when Γ is trivial, S_{n^2} is monoidally equivalent with the so-called even part of O_n^+. We also obtained preliminary results concerning the $\mathbb{H}$-intertwiner spaces using Proposition 3.2.15 of this thesis describing intertwiner spaces in the free products of certain compact quantum groups. Similar arguments also hold for the more general free wreath products $\mathbb{G}\wr_* S_{n^2}^+$.

Chapter 2

Haagerup approximation property for quantum reflection groups

This chapter is the text of an article published in Proceedings of the American Mathematical Society [Lem13b]. We prove that the duals of the quantum reflection groups H_N^{s+} have the Haagerup property for all $N \geq 4$ and $s \in [1, \infty)$. We use the canonical arrow $\pi : C(H_N^{s+}) \to C(S_N^+)$ onto the quantum permutation groups, and we describe how the characters of $C(H_N^{s+})$ behave with respect to this morphism π thanks to the description of the fusion rules binding irreducible corepresentations of $C(H_N^{s+})$ ([BV09]). This allows us to construct states on the central C^*-algebra $C(H_N^{s+})_0$ generated by the characters of $C(H_N^{s+})$ and to use a fundamental theorem proved by M.Brannan giving a method to construct nets of trace-preserving, normal, unital and completely positive maps on the von Neumann algebra of a compact quantum group $\mathbb{G}$ of Kac type ([Bra12a]).

Introduction

A (classical) discrete group Γ has the Haagerup property if (and only if) there is a net (φ_i) of normalized positive definite functions in $C_0(\Gamma)$ converging pointwise to the constant function 1. There are lots of examples of discrete groups with the Haagerup property: all amenable groups have this property. The free groups F_N are examples of discrete groups with Haagerup property (see [Haa79]) but which are not amenable. Thus, one says that the Haagerup property is a weak form of amenability. This property is also known as a "strong negation" of Kazhdan's property (T): the only (classical) discrete groups with both properties are finite. Another weak form of amenability is the weak amenability, see below for examples in the quantum setting. One can find more examples and a more complete approach to the problems and questions related to the Haagerup property, also called “a-T-amenability", in [CCJ$^+$].

The Haagerup property has many interests in various fields of mathematics such as geometry of groups or functional analysis. We can mention e.g. groups with wall space structures (see [CSV12] and [CDH10]) as illustrations of the interest in the Haagerup property with respect to the theory of geometry of groups. In functional analysis, the Haagerup property appears e.g. in questions related to the Baum-Connes conjecture (see [HK01]) or in Popa's deformation/rigidity techniques (see [Pop06]).

In [Bra12a], a natural definition of the Haagerup property for compact quantum groups $\mathbb{G}$ of Kac type is proposed: $\mathbb{G}$ has the Haagerup approximation property if and only if its associated (finite) von Neumann algebra $L^\infty(\mathbb{G})$ has the Haagerup property (see also below, Definition 2.1.1). We use this definition with the slight modification: the *dual* $\widehat{\mathbb{G}}$ of $\mathbb{G}$ has the Haagerup property if $L^\infty(\mathbb{G})$ has the Haagerup property, so that this definition is closer to the classical case where $\widehat{\mathbb{G}}$ is a classical discrete group. The author of [Fre13] proposes another definition for the Haagerup property of discrete quantum groups: $\widehat{\mathbb{G}}$ has the Haagerup property if there exists a net (a_i) in $c_0(\widehat{\mathbb{G}})$ which converges to 1 pointwise and such that the associated multipliers m_{a_i} are unital and completely positive. These approaches are equivalent in the unimodular case. We refer the reader to [DFSW13] for more informations on the Haagerup property in the (more general) context of locally compact quantum groups.

In [Bra12a] and [Bra12b], the author shows that the duals of the compact quantum groups O_N^+, U_N^+ and S_N^+, introduced by Wang (see [Wan95] and [Wan98]) have the Haagerup property. In fact, in [Bra12b], it is proved that any trace-preserving quantum automorphism group of a finite dimensional C^*-algebra has the Haagerup property. In [Fre13], using some block decompositions and Brannan's proof of the fact that $\widehat{O_N^+}$ has the Haagerup property (precisely that some completely positive multipliers can be found), the author proves that $\widehat{O_N^+}$ is weakly amenable (in fact, it is also proved in [Fre13] that the $\widehat{U_N^+} \subset \mathbb{Z} * \widehat{O_N^+}$ is weakly amenable, and an argument of monoidal equivalence allows to prove, in particular, that $\widehat{S_N^+}$ is weakly amenable too). In [Fim10], a definition of property (T) for discrete quantum groups and some classical properties for discrete groups are generalized, for instance: discrete quantum groups with property (T) are finitely generated and unimodular.

The aim of this chapter is to prove that the duals of quantum reflection groups H_N^{s+}, introduced in [BBCC11], have the Haagerup property. It is a natural generalization of the case $s = 1$ treated in [Bra12b] (since $H_N^{1+} = S_N^+$). However, this generalization is not immediate: as a matter of fact, the sub C^*-algebra generated by the characters is not commutative so that the strategy used in [Bra12a] and [Bra12b] does not work anymore.

However, a fundamental tool of the proof of the main result of this chapter is [Bra12a, Theorem 3.7].

What motivates this chapter of our thesis is also the fact that quantum reflection groups are free wreath products between $\mathbb{Z}_s$ and S_N^+ (see [Bic04] and Theorem 2.1.14 below) and the result we prove in this chapter naturally leads to the following question: is it true that if Γ is a discrete group which has the Haagerup property then $\widehat{\Gamma} \wr_w S_N^+$ has the Haagerup property ? One can notice the similarity with the result in [CSV12] concerning (classical) wreath products of discrete groups: If Γ, Γ' are countable discrete groups with the Haagerup property then $\Gamma \wr \Gamma'$ also has the Haagerup property. This similarity is however formal: in this thesis we are considering (free) wreath products of groups whose *duals* have the Haagerup property.

Our proof of the fact $\widehat{H_N^{s+}}$ have the Haagerup property relies on the knowledge of the fusion rules of the associated compact quantum group H_N^{s+}, determined in [BV09]. Indeed, there is no general result about fusion rules for free wreath products of compact quantum groups yet.

The rest of this chapter is organized as follows. In the section 2.1, we recall the definition of the Haagerup property for compact quantum groups of Kac type and we give the result of Brannan concerning the construction of normal, unital, completely positive and trace-preserving maps on $L^\infty(\mathbb{G})$ (see Theorem 1.2.10). We also give a positive answer to a question asked in [Wor87a], in the discrete and Kac setting case, concerning symmetric tensors with respect to the coproduct. Then we collect some results on Tchebyshev polynomials: some are already mentioned and used in [Bra12a], but we give suitably adapted statements and proofs for our purpose. Thereafter, we recall the definition of quantum reflection groups H_N^{s+}, and we describe their irreducible corepresentations and the fusion rules binding them. We also recall that at $s = 1$, we get the quantum permutation groups S_N^+. In section 2.2, we identify the images of the irreducible characters of $C(H_N^{s+})$ by the canonical morphism onto $C(S_N^+)$. In the section 2.3, we prove that the duals of the quantum reflection groups H_N^{s+} have the Haagerup approximation property for all $N \geq 4$.

2.1 Preliminaries

Let us first fix some notations. One can refer to [Bra12a], [VV07], [KV00] and [Wor98] for more details. Recall that $\mathbb{G} = (C(\mathbb{G}), \Delta)$ will denote a compact quantum group, where $C(\mathbb{G})$ is a full Woronowicz C^*-algebra. Furthermore, every compact quantum group $\mathbb{G}$ considered in this chapter is of Kac type (or equivalentely, its dual $\widehat{\mathbb{G}}$ is unimodular) that is: the unique Haar state h on $C(\mathbb{G})$, is tracial. (We recall that $L^\infty(\mathbb{G})$ is defined by $L^\infty(\mathbb{G}) = C_r(\mathbb{G})'' = \lambda_h(C(\mathbb{G}))''$, where $(L^2(\mathbb{G}), \lambda_h)$ is the GNS construction associated to h).

2.1.1 Haagerup property for compact quantum groups of Kac type

Definition 2.1.1. *The dual $\widehat{\mathbb{G}}$ of a compact quantum group $\mathbb{G} = (C(\mathbb{G}), \Delta)$ of Kac type has the Haagerup approximation property if the finite von Neumann algebra $(L^\infty(\mathbb{G}), h)$ has the Haagerup approximation property i.e. if there exists a net (ϕ_x) of trace preserving, normal, unital and completely positive maps on $L^\infty(\mathbb{G})$ such that their unique extensions to $L^2(\mathbb{G})$ are compact operators and (ϕ_x) converges to $id_{L^\infty(\mathbb{G})}$ pointwise in L^2-norm.*

One essential tool to construct nets of normal, unital, completely positive and trace preserving maps (we will say *NUCP trace preserving* maps) is the next theorem proved in [Bra12a]. We will denote by $Irr(\mathbb{G})$ the set indexing the equivalence classes of irreducible corepresentations of a compact quantum group $\mathbb{G}$ and by $Pol(\mathbb{G})$ the linear space spanned by the matrix coefficients of such corepresentations $u^\alpha, \alpha \in Irr(\mathbb{G})$. If $\alpha \in Irr(\mathbb{G})$, let $L^2_\alpha(\mathbb{G}) \subset L^2(\mathbb{G})$ be the subspace spanned by the GNS images of matrix coefficients $u^\alpha_{ij},\ i,j \in \{1,\dots,d_\alpha\}$ of the irreducible unitary corepresentation u^α ($d_\alpha = \dim(u^\alpha_{ij})$) and $p_\alpha : L^2(\mathbb{G}) \to L^2_\alpha(\mathbb{G})$ be the associated orthogonal projection. Then $L^2(\mathbb{G}) = l^2 - \bigoplus_{\alpha\in Irr(\mathbb{G})} L^2_\alpha(\mathbb{G})$. We denote by $C(\mathbb{G})_0 \subset C(\mathbb{G})$ the C^*-algebra generated by the irreducible characters $\chi_\alpha = \sum_{i=1}^{d_\alpha} u^\alpha_{ii}$ of a compact quantum group $\mathbb{G}$ and $\chi_{\overline{\alpha}}$ the character of the associated conjugate corepresentation $u^{\overline{\alpha}}$.

We will use in an essential way Theorem 1.2.10 which provides a construction of NUCP maps on $L^\infty(\mathbb{G})$ starting from states on the central algebra of a compact quantum group of Kac type.

The averaging methods used to prove this theorem allow us to answer, in a restricted setting, a question asked in [Wor87a].

Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group. Then consider the C^*-subalgebra $C(\mathbb{G})_{\text{central}} := \{a \in C(\mathbb{G}) : \Delta(a) = \Sigma \circ \Delta(a)\}$ i.e. the C^*-subalgebra of the symmetric tensors in $C(\mathbb{G}) \otimes C(\mathbb{G})$ with respect to Δ (Σ denotes the usual flip map $\Sigma : C(\mathbb{G}) \otimes C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G}), a\otimes b \mapsto b \otimes a$). In [Wor87a], the author also defines $Pol(\mathbb{G})_{\text{central}} := \{a \in Pol(\mathbb{G}) : \Delta(a) = \Sigma \circ \Delta(a)\}$. We recall the question asked by Woronowicz (see [Wor87a] thereafter Proposition 5.11):

Question 2.1.2. *Is $Pol(\mathbb{G})_{\text{central}}$ dense in $C(\mathbb{G})_{\text{central}}$ (for the norm of $C(\mathbb{G})$) ?*

Then the answer is yes, at least in the Kac and discrete setting. We simply denote by $||.||$ the norm on $C(\mathbb{G})$. It is clear, and proved in [Wor87a], that $Pol(\mathbb{G})_{\text{central}} = \text{span}\{\chi_\alpha : \alpha \in Irr(\mathbb{G})\}$ where $\chi_\alpha = \sum_{i=1}^{d_\alpha} u_{ii}^\alpha$ denotes the character of an irreducible finite dimentional corepresentation (u_{ij}^α). So the problem reduces to prove that $C(\mathbb{G})_{\text{central}} \subset \overline{\text{span}}^{||\cdot||}\{\chi_\alpha : \alpha \in Irr(\mathbb{G})\}$, the other inclusion being clear.

Theorem 2.1.3. *Let $\mathbb{G}_r = (C(\mathbb{G}_r), \Delta_r)$ be a compact quantum group of Kac type with faithful Haar state. Then $\overline{Pol(\mathbb{G}_r)_{\text{central}}}^{||\cdot||} = C(\mathbb{G}_r)_{\text{central}}$.*

Proof. We first note that Δ_r preserves the trace in the sense that $(h \otimes h) \circ \Delta_r = h$. As a result the Hilbertian adjoint Δ_r^*, of the L^2-extension of Δ_r, is well-defined and we have $||\Delta_r^*(x)|| \leq ||x||$ for $x \in C(\mathbb{G}_r) \otimes C(\mathbb{G}_r)$ with respect to the operator norms (note that this is particular to the tracial situation). Since Δ_r^* clearly maps the subspace $Pol(\mathbb{G}_r) \otimes Pol(\mathbb{G}_r)$ of $L^2(\mathbb{G}_r) \otimes L^2(\mathbb{G}_r)$ to $Pol(\mathbb{G}_r)$, it also restricts to a contractive map from $C(\mathbb{G}_r) \otimes C(\mathbb{G}_r)$ to $C(\mathbb{G}_r)$, still denoted Δ_r^*. Now we put $E = \Delta_r^* \circ \Sigma \circ \Delta_r : C(\mathbb{G}_r) \to C(\mathbb{G}_r)$. We have $||E|| \leq 1$, and for $a \in C(\mathbb{G}_r)_{\text{central}}, E(a) = \Delta_r^* \circ \Delta_r(a) = a$ so that $C(\mathbb{G}_r)_{\text{central}} \subset E(C(\mathbb{G}_r))$.

But, on the other hand, for any matrix coefficient of a finite dimensional unitary corepresentations (u_{ij}^α), we have

$$E(u_{ij}^\alpha) = \Delta_r^* \circ \Sigma \circ \Delta_r(u_{ij}^\alpha) = \Delta_r^* \circ \Sigma \left(\sum_k u_{ik}^\alpha \otimes u_{kj}^\alpha \right) = \Delta_r^* \left(\sum_k u_{kj}^\alpha \otimes u_{ik}^\alpha \right).$$

We compute $\Delta_r^* \left(\sum_k u_{kj}^\alpha \otimes u_{ik}^\alpha \right)$ using the duality pairing induced by the inner product coming from the Haar state h: let $\beta \in Irr(\mathbb{G}_r)$, then for all $1 \leq p, q \leq d_\beta$:

$$\begin{aligned} \left\langle u_{pq}^\beta, \Delta_r^* \left(\sum_k u_{kj}^\alpha \otimes u_{ik}^\alpha \right) \right\rangle_h &= \sum_{l,k} \left\langle u_{pl}^\beta \otimes u_{lq}^\beta, u_{kj}^\alpha \otimes u_{ik}^\alpha \right\rangle_h = \frac{\delta_{\alpha\beta}\delta_{ij}\delta_{pq}}{d_\alpha^2} \\ &= \left\langle u_{pq}^\beta, \frac{\delta_{ij}}{d_\alpha} \chi_\alpha \right\rangle_h . \end{aligned}$$

Then summarizing, we have $E(u_{ij}^\alpha) = \frac{\delta_{ij}}{d_\alpha} \chi_\alpha \in \mathrm{Pol}(\mathbb{G}_r)_{\text{central}}$, $||E|| = 1$ and $E|_{Pol(\mathbb{G}_r)_{\text{central}}} = id$. Thus we obtain a conditional expectation $E : C(\mathbb{G}_r) \to \overline{Pol(\mathbb{G}_r)_{\text{central}}}^{||\cdot||} = \overline{\text{span}}^{||\cdot||}\{\chi_\alpha : \alpha \in Irr(\mathbb{G}_r)\}$. But we have $C(\mathbb{G}_r)_{\text{central}} \subset E(C(\mathbb{G}_r))$ and the result follows. ☐

Notation 2.1.4. *We will denote by $Pol(\mathbb{G})_0$ and $C(\mathbb{G})_0$ the central $*$-algebras and C^*-algebras generated by the irreducible characters of a compact quantum group $\mathbb{G}$.*

2.1.2 Tchebyshev polynomials.

Definition 2.1.5. *We define a family of polynomials $(A_t)_{t\in\mathbb{N}}$ as follows: $A_0 = 1, A_1 = X$ and for all $t \geq 1$*

$$A_1 A_t = A_{t+1} + A_{t-1}. \tag{2.1}$$

We call them the dilated Tchebyshev polynomials of second kind.

We will use the following results on Tchebyshev polynomials A_t. The second one is based upon a result proved in [Bra12a, Proposition 4.4], but suitably adapted to our purpose.

Proposition 2.1.6. *for all $t, s \geq 1$ we have: $A_t A_s = A_{t+s} + A_{t-1}A_{s-1}$*

Proof. This result is easily proved by induction on $t \geq 1$. □

Proposition 2.1.7. *Let $N \geq 2$. For all $x \in (2, N)$, there exists a constant $c \in (0, 1)$ such that for all integers $t \geq 1$ we have*

$$0 < \frac{A_t(x)}{A_t(N)} \leq \left(\frac{x}{N}\right)^{ct}.$$

Proof. First, we follow the proof of [Bra12a, Proposition 4.4] and introduce the function $q(x) = \frac{x+\sqrt{x^2-4}}{2}$, for $x > 2$. Then an induction and the recursion formula (2.1) for the polynomials A_t show that for all $t \geq 0$, we have

$$A_t(x) = \frac{q(x)^{t+1} - q(x)^{-t-1}}{q(x) - q(x)^{-1}}$$

Then using the same tricks as in [Bra12a], we get that for all fixed $x \in (2, N)$ and all $t \geq 1$

$$\begin{aligned}\frac{A_t(x)}{A_t(N)} &= \frac{q(x)^{t+1} - q(x)^{-t-1}}{q(N)^{t+1} - q(N)^{-t-1}} \frac{q(N) - q(N)^{-1}}{q(x) - q(x)^{-1}} \\ &= \left(\frac{x}{N}\right)^t \left(\frac{1 + \sqrt{1 - \frac{4}{x^2}}}{1 + \sqrt{1 - \frac{4}{N^2}}}\right)^t \frac{1 - q(x)^{-2t-2}}{1 - q(N)^{-2N-2}} \frac{1 - q(N)^{-2}}{1 - q(x)^{-2}}.\end{aligned}$$

Now notice that the factor $\frac{1-q(x)^{-2t-2}}{1-q(N)^{-2N-2}}$ is less than 1 because q is increasing. Furthermore, we have

$$\left(\frac{1 + \sqrt{1 - \frac{4}{x^2}}}{1 + \sqrt{1 - \frac{4}{N^2}}}\right)^t \frac{1 - q(N)^{-2}}{1 - q(x)^{-2}} \underset{t\to\infty}{\longrightarrow} 0$$

since the last factor does not depend on t and $\frac{1+\sqrt{1-\frac{4}{x^2}}}{1+\sqrt{1-\frac{4}{N^2}}} < 1$. Hence, there exists t_0 such that $\dfrac{A_t(x)}{A_t(N)} \leq \left(\dfrac{x}{N}\right)^t$ for all $t \geq t_0$. It remains to show that there exists $c \in (0,1)$ such that $\dfrac{A_t(x)}{A_t(N)} \leq (\frac{x}{N})^{ct_0}$ for all $t = 1, \ldots, t_0 - 1$, since for all $0 < t < t_0$, $\left(\dfrac{x}{N}\right)^{ct_0} \leq \left(\dfrac{x}{N}\right)^{ct}$. To prove that such a c exists, we notice that $\max\left\{\frac{A_t(x)}{A_t(N)} : t = 1, \ldots, t_0 - 1\right\} := D < 1$ since the Tchebyshev polynomials are increasing on $(2, +\infty)$. Hence, it is clear that we can find $c > 0$ such that $\left(\dfrac{x}{N}\right)^{ct_0} \geq D$. □

Remark 2.1.8. *The following hold.*

1. *In [Bra12a, Proposition 6.4], the exponent is better (there is no constant c) but there is a constant multiplying $\left(\dfrac{x}{N}\right)^t$. Our version allows an easy proof of Proposition 2.3.3 below.*

2. *The previous proposition gives information on the behavior of the dilated Tchebyshev polynomials on $(2, +\infty)$: the quotient $\dfrac{A_t(x)}{A_t(N)}$ has an exponential decay with respect to $t \geq 1$. We will also need some informations on this quotient when $x \in (0, 2)$ and $N = 2$. That is the aim of the next paragraph.*

The polynomials A_t are linked to the Tchebyshev polynomials of second kind U_t by the following formula: $\forall t \in \mathbb{N}, x \in [0,1], A_t(2x) = U_t(x)$. Indeed, we recall (see [Riv] for more details) that the Tchebyshev polynomials of second kind U_t are defined for all $x \in [-1,1]$ by

$$U_t(x) = \frac{\sin((t+1)\arccos(x))}{\sqrt{1-x^2}} = \frac{\sin((t+1)\theta)}{\sin(\theta)}, \quad \text{with } x = \cos(\theta). \tag{2.2}$$

In particular, $U_0 = 1$, $U_1(x) = 2x$ and for all $t \in \mathbb{N}^*, U_t(1) = t + 1$. Then one can check that for all $t \in \mathbb{N}$ and $x \in [0,1]$: $2xU_t(x) = U_{t+1}(x) + U_{t-1}(x)$.

Proposition 2.1.9. *Let $x \in (0, 2)$. Then for any integer $t \geq 1$*

$$\left|\frac{A_t(x)}{A_t(2)}\right| = \frac{1}{t+1}\left|\frac{\sin((t+1)\theta)}{\sin(\theta)}\right|, \quad \textit{with } x = \cos(\theta).$$

In particular, there exists a positive constant $D < 1$ such that $\forall t \geq 1$, $\left|\dfrac{A_t(x)}{A_t(2)}\right| \leq D$ and $\dfrac{A_t(x)}{A_t(2)} \longrightarrow 0$ as $t \to \infty$.

Proof. First, by what we recalled above, we can write $\frac{A_t(x)}{A_t(2)} = \frac{U_t(\frac{x}{2})}{U_t(1)} = \frac{U_t(\frac{x}{2})}{t+1}$. Thus, if $x = 2\cos(\theta)$, we have by the relation (2.2) above

$$\left|\frac{A_t(x)}{A_t(2)}\right| = \frac{1}{t+1}\left|\frac{\sin((t+1)\theta)}{\sin(\theta)}\right| \underset{t\to\infty}{\longrightarrow} 0.$$

On the other hand, on $[0,1]$, the polynomials $U_t, t \geq 1$ have $t+1$ as a maximum, only attained in 1. Then, it is clear that for all $t \geq 1$ and $x \in (0,2)$:

$$0 < \frac{A_t(x)}{A_t(2)} = \frac{U_t(\frac{x}{2})}{t+1} < 1.$$

So the existence of the announced constant D is clear. □

2.1.3 Quantum reflection groups

In this subsection, we recall the definition of the quantum reflection groups H_N^{s+} and the particular case of the quantum permutation groups S_N^+. We also recall that $C(H_N^{s+})$ is the free wreath product of two quantum permutation algebras. In the end of this subsection, we recall the description of the irreducible corepresentations of $C(H_N^{s+})$ together with the fusion rules binding them.

Recall from Definition 1.1.17 that for all $s \geq 1$ and $N \geq 2$, the quantum reflection group H_N^{s+} is the pair $(C(H_N^{s+}), \Delta)$ composed of the universal C^*-algebra generated by N^2 normal elements U_{ij} satisfying the following relations $U = (U_{ij})$ is unitary, ${}^tU = (U_{ji})$ is unitary, $p_{ij} = U_{ij}U_{ij}^*$ is a projection, $U_{ij}^s = p_{ij}$ and such that U is a corepresentation of $C(H_N^{s+})$.

Remark 2.1.10.

1. *For $s = 1$ we get the quantum permutation group S_N^+. The definition of S_N^+ thus may be summed up as follows (see also [Wan98]): S_N^+ is the pair $(C(S_N^+), \Delta)$ where*

   (a) *$C(S_N^+)$ is the universal C^*-algebra generated by N^2 elements v_{ij} such that the matrix $v = (v_{ij})$ is unitary and $v_{ij} = v_{ij}^* = v_{ij}^2$ (i.e. v is a magic unitary).*

   (b) *The coproduct is given by the usual relations making of v a corepresentation (the fundamental one) of $C(S_N^+)$.*

2. *For $s = 2$, we find the hyperoctahedral quantum group, i.e. the easy quantum group H_N^+ studied e.g. in [Web13].*

3. *There is a morphism $C(H_N^{s+}) \to C(S_N^+)$ of compact quantum groups: one only has to check that the generators v_{ij} of $C(S_N^+)$, satisfy the relations described in Definition 1.1.17, which is clear.*

Notation 2.1.11. *We will denote by $\pi : C(H_N^{s+}) \to C(S_N^+)$ the canonical arrow mentioned in the remark above.*

We refer to Theorem 1.1.13 for the results concerning the irreducible corepresentations of $C(S_N^+)$. We denote by $\chi_k = \sum_{i=1}^{d_k} v_{ii}^{(k)}$ the character associated to $v^{(k)}$.

We will need the following proposition, proved in [Bra12b]:

Proposition 2.1.12. *Let χ be the character associated to the fundamental corepresentation v of $C(S_N^+)$. Then, $\chi^* = \chi$ and there is a $*$-isomorphism $C^*(\chi) = C(S_N^+)_0 = C^*(\chi_t : t \in \mathbb{N}) \simeq C([0,N])$ identifying χ_t to the polynomial defined Π_t by $\Pi_0 = 1, \Pi_1 = X - 1$ and $\forall t \geq 1, \Pi_1\Pi_t = \Pi_{t+1} + \Pi_t + \Pi_{t-1}$.*

Remark 2.1.13. *Notice that:*

1. *The recursion formula defining the polynomials Π_t is the one satisfied by the irreducible characters χ_t.*

2. *The polynomials A_t and Π_t are linked by the formula: $\Pi_t(x) = A_{2t}(\sqrt{x})$.*

Before describing the fusion rules of $C(H_N^{s+})$, we recall that these compact quantum groups are free wreath products:

Theorem 2.1.14. *[BV09, Theorem 3.4] Let $N \geq 2$, then we have the following isomorphisms of compact quantum groups:*

- *$C(H_N^{s+}) \simeq C(\mathbb{Z}_s) *_w C(S_N^+) = C^*(\mathbb{Z}_s^{*N}) * C(S_N^+)/ < [z_i, v_{ij}] = 0 >$ where z_i is the generator of the i-th copy $\mathbb{Z}_s$ in the free product $\mathbb{Z}_s^{*N}$.*

- *In particular $C(H_2^{s+}) \simeq C(\mathbb{Z}_s) *_w C(Z_2)$, $C(H_3^{s+}) \simeq C(\mathbb{Z}_s) *_w C(S_3)$.*

We refer to Theorem 1.1.18 for the description of the irreducible corepresentations of $C(H_N^{s+})$.

Notation 2.1.15. *We will denote the basic irreducible corepresentations of $C(H_N^{s+})$ by $\rho_t, t \in \{0, \dots, s-1\}$, with $\rho_t = U_t$ $\forall t \in \{1, \dots, s-1\}$ and $\rho_0 = U_0 \ominus 1$ (where $U_0 = (U_{ij}U_{ij}^*)$).*

The proof of the first three assertions follows from the definitions of corepresentations of compact quantum groups and of the definition of $C(H_N^{s+})$. The proof of the last three assertions is based upon Woronowicz's Tannaka-Krein duality (see [Wor88]) and methods inspired by [Ban96], [Ban99b] and [BBCC11]. Now, we can give the description of the fusion rules:

Theorem 2.1.16. *[BV09, Theorem 8.2] Let M be the monoid $M = \langle a, z : z^s = 1\rangle$ with involution $a^* = a$, $z^* = z^{-1}$, and the fusion rules obtained by recursion from the formulae*

$$vaz^i \otimes z^j aw = vaz^{i+j}aw \oplus \delta_{i+j,0}\,(v \otimes w) \tag{2.3}$$

Then the irreducible corepresentations r_α of $C(H_N^{s+})$ can be indexed by the elements α of the submonoid S generated by the elements $az^ia, i = 0, \dots, s-1$, with involution and fusion rules above.

Remark 2.1.17.

1. *S is composed of elements $a^{L_1}z^{J_1}\dots z^{J_{K-1}}a^{L_K}$ with*
   - *$J_i, L_i > 0$ integers.*
   - *L_1, L_K odd integers and all the L_i's, $i \in \{2, \dots, K-1\}$ even integers.*
   - *Except if $K = 1$, then L_K is an even integer.*
2. *With this description, we can identify the basic corepresentations introduced above: the corepresentation r_{a^2} is the corepresentation $\rho_0 = (U_{ij}U_{ij}^*) \ominus 1$ and for $t \neq 0$, r_{az^ta} is the corepresentation $\rho_t = (U_{ij}^t)$.*
3. *In Proposition 2.2.1, we will use the suggestive notation*

   $$vaz^{i+j}aw = (vaz^i \otimes z^j aw) \ominus \delta_{i+j,0}(v \otimes w),$$

   which simply means that we have the relation (2.3) in the monoid S.
4. *If $\alpha = a^{L_1}z^{J_1}\dots z^{J_{K-1}}a^{L_K} \in S$, then the conjugate corepresentation of r_α is indexed by $\overline{\alpha} = a^{L_K}z^{-J_{K-1}}\dots z^{-J_1}a^{L_1}$*

We end this subsection by the following proposition which summarizes the results above:

Proposition 2.1.18. *The canonical morphism $\pi : C(H_N^{s+}) \to C(S_N^+)$ maps all the corepresentations $U_t, t \in \mathbb{Z}$ onto the fundamental corepresentation v of $C(S_N^+)$; in other words, it maps all $\rho_t = r_{az^ta}, t \neq 0$ onto v and $\rho_0 = r_{a^2}$ onto $v^{(1)}$.*

2.2 Characters of quantum reflection groups and quantum permutation groups

As announced in the introduction, we find the images of the irreducible characters of $C(H_N^{S+})$ under the canonical morphism $\pi : C(H_N^{s+}) \to C(S_N^+)$.

Proposition 2.2.1. *Let χ_α be the character of an irreducible corepresentation r_α of $C(H_N^{s+})$. Write $\alpha = a^{l_1} z^{j_1} \dots z^{j_{k-1}} a^{l_k}$. Then, identifying $C(S_N^+)_0$ with $C([0,N])$, the image of χ_α, say P_α, satisfies:*

$$P_\alpha(X^2) = \pi(\chi_\alpha)(X^2) = \prod_{i=1}^{k} A_{l_i}(X).$$

Proof. We shall prove this proposition by induction on the even integer $\sum_{i=1}^k l_i$ using the description of the fusion rules given by Theorem 2.1.16, the recursion formula satisfied by the Tchebyshev polynomials, Proposition 2.1.6 and Proposition 2.1.18.

Let HR(λ) be the following statement: $\pi(\chi_\alpha)(X^2) = \prod_{i=1}^k A_{l_i}(X)$ for any $\alpha = a^{l_1} z^{j_1} \dots z^{j_{k-1}} a^{l_k}$ such that $2 \leq \sum_i l_i \leq \lambda$.

Let us begin by studying simple examples (and initializing the induction).

Consider the element aza. Then, the irreducible corepresentation r_{aza} (written ρ_1 in Notation 2.1.15) is sent by π onto $v = 1 \oplus v^{(1)}$ by Proposition 2.1.18. Thus, in term of characters, we obtain by Proposition 2.1.12

$$\pi(\chi_{aza})(X) = 1 + (X-1) = X = A_1(X)$$

i.e.

$$P_{aza}(X^2) = X^2 = A_1(X)A_1(X).$$

Actually, this holds for all elements $\alpha = az^j a,\ j \in \{1, \dots, s-1\}$ (since every irreducible corepresentation r_{az^ja} is sent by π onto $1 \oplus v^{(1)}$, as is r_{aza}).

Consider the element a^2. Then, the irreducible corepresentation r_{a^2} (written ρ_0 in Notation 2.1.15) is sent by π onto $v^{(1)}$. Thus $\pi(\chi_{a^2})(X) = X - 1$. i.e.

$$P_{a^2}(X^2) = X^2 - 1 = A_2(X).$$

To prove HR(2) one has to show that $\pi(\chi_{a^2})(X^2) = A_2(X)$ and $\pi(\chi_{az^ja})(X^2) = A_1 A_1(X)$ for all $j \in \{1, \dots, s-1\}$, what we have just done above.

Now assume HR(λ) holds: $\pi(\chi_\beta)(X^2) = \prod_{i=1}^k A_{l_i}(X)$ for any $\beta = a^{l_1} z^{j_1} \dots z^{j_{k-1}} a^{l_k}$ such that $2 \leq \sum_i l_i \leq \lambda$. We now show HR($\lambda + 2$).

Let $\alpha = a^{L_1} z^{J_1} \dots a^{L_K}$, with $\sum_i L_i = \lambda + 2$. In order to use HR(λ), we must "break" α using the fusion rules as in the examples above. Then, essentially, one has to distinguish the cases $L_K = 1, L_K = 3$ and $L_K \geq 5$ (in the case $L_K \geq 5$ we can "break α at a^{L_K}" but in the other cases we must use $a^{L_{K-1}}$ or $a^{L_{K-2}}$ if they exist, that is if there are enough factors a^{L_i}). So first, we deal with two special cases below, in order to have "enough" factors a^L in α in the sequel. We use the fusion rules described in Theorem 2.1.16 (and the notations described after, see Remark 2.1.17).

- If $K = 1$ i.e. $L_K = \lambda + 2,\ J_i = 0\ \forall i$, write:

$$\alpha = a^{\lambda+2} = (a^\lambda \otimes a^2) \ominus (a^{\lambda-1} \otimes a) = (a^\lambda \otimes a^2) \ominus a^\lambda \ominus a^{\lambda-2}.$$

  Then using the hypothesis of induction and Proposition 2.1.6, we get

$$\begin{aligned}
\pi(\chi_\alpha)(X^2) &= A_\lambda A_2(X) - A_\lambda(X) - A_{\lambda-2}(X) \\
&= A_\lambda A_2(X) - (A_\lambda(X) + A_{\lambda-2}(X)) \\
&= A_\lambda A_2(X) - A_{\lambda-1} A_1(X) \\
&= A_{\lambda+2}(X).
\end{aligned}$$

  (Notice that if $\lambda = 2$ one has $\lambda - 2 = 0$ and $a^4 = (a^2 \otimes a^2) \ominus (a \otimes a) = (a^2 \otimes a^2) \ominus a^2 \ominus 1$ so that the result we want to prove then is still true.)

- If $K = 2, J := J_1 \neq 0$, write $\alpha = a^{L_1} z^J a^{L_2}$. We have $L_1 + L_2 = \lambda + 2 \geq 4$ and L_1, L_2 are odd hence L_1 or $L_2 \geq 3$, say $L_1 \geq 3$. Write

$$a^{L_1} z^J a^{L_2} = (a^2 \otimes a^{L_1 - 2} z^J a^{L_2}) \ominus (a \otimes a^{L_1 - 3} z^J a^{L_2}).$$

  If $L_1 = 3$ then the tensor product $a \otimes a^{L_1-3} z^J a^{L_2}$ is equal to $a z^J a^{L_2}$ hence $\alpha = a^3 z^J a^{L_2}$ satisfies

$$\begin{aligned}
\pi(\chi_\alpha)(X^2) &= A_2 A_1 A_{L_2}(X) - A_1 A_{L_2}(X) \\
&= A_3(X) A_{L_2}(X).
\end{aligned}$$

If $L_1 > 3$ (i.e. $L_1 \geq 5$), then the tensor product $a \otimes a^{L_1-3} z^J a^{L_2}$ is equal to $a^{L_1-2} z^J a^{L_2} \oplus a^{L_1-4} z^J a^{L_2}$. We get

$$\begin{aligned}\pi(\chi_\alpha)(X^2) &= A_2 A_{L_1-2} A_{L_2}(X) - A_{L_1-2} A_{L_2}(X) - A_{L_1-4} A_{L_2}(X)\\ &= A_{L_1}(X) A_{L_2}(X).\end{aligned}$$

- From now on, we suppose that there are more than three factors a^{L_i} in α i.e. $K \geq 3$. We will have to distinguish three cases: $L_K = 1, L_K = 3$ and $L_K \geq 5$.

  If $5 \leq L_K < \sum_i L_i$, write $L_K = m_K + 2$. Then we have $m_K \geq 3$, so

$$\begin{aligned}a^{L_1} z^{J_1} \ldots a^{L_K} &= a^{L_1} z^{J_1} \ldots a^{m_K+2}\\ &= (a^{L_1} z^{J_1} \ldots a^{m_K} \otimes a^2) \ominus (a^{L_1} z^{J_1} \ldots a^{m_K-1} \otimes a)\\ &= (a^{L_1} z^{J_1} \ldots a^{m_K} \otimes a^2) \ominus a^{L_1} z^{J_1} \ldots a^{m_K} \ominus a^{L_1} z^{J_1} \ldots a^{m_K-2}.\end{aligned}$$

  Then

$$\begin{aligned}\pi(\chi_\alpha)(X^2) &= A_{L_1} \ldots A_{L_{K-1}} A_{m_k} A_2(X) - A_{L_1} \ldots A_{m_K}(X) - A_{L_1} \ldots A_{m_K-2}(X)\\ &= A_{L_1} \ldots A_{L_{K-1}} A_{L_K}(X).\end{aligned}$$

  If $m_K = 1$, i.e. $L_K = 3$, we proceed in the same way using

$$a^{L_1} z^{J_1} \ldots z^{J_{K-1}} a^3 = (a^{L_1} z^{J_1} \ldots a \otimes a^2) \ominus a^{L_1} z^{J_1} \ldots z^{J_{K-1}} a.$$

  To conclude the induction, one has to deal with the case $L_K = 1$. We have to distinguish the following cases:

  If $L_{K-1} \geq 4$. We have

$$\begin{aligned}a^{L_1} z^{J_1} \ldots a^{L_{K-1}} z^{J_{K-1}} a &= (a^{L_1} z^{J_1} \ldots a^{L_{K-1}-1} \otimes a z^{J_{K-1}} a) \ominus (a^{L_1} z^{J_1} \ldots a^{L_{K-1}-2} \otimes z^{J_{K-1}} a)\\ &= (a^{L_1} z^{J_1} \ldots a^{L_{K-1}-1} \otimes a z^{J_{K-1}} a) \ominus a^{L_1} z^{J_1} \ldots a^{L_{K-1}-2} z^{J_{K-1}} a.\end{aligned}$$

Then

$$\begin{aligned}\pi(\chi_\alpha)(X^2) &= A_{L_1}\dots A_{L_{K-1}-1}A_1A_1(X) - A_{L_1}\dots A_{L_{K-1}-2}A_1(X)\\ &= A_{L_1}\dots A_{L_{K-1}}A_1(X).\end{aligned}$$

If $L_{K-1}=2$ and $J_{K-1}+J_{K-2}=0$ mod s, we can proceed in the same way using

$$\begin{aligned}&a^{L_1}z^{J_1}\dots a^{L_{K-2}}z^{J_{K-2}}a^2z^{J_{K-1}}a\\ &\quad = (a^{L_1}z^{J_1}\dots a^{L_{K-2}}z^{J_{K-2}}a\otimes az^{J_{K-1}}a)\ominus a^{L_1}z^{J_1}\dots a^{L_{K-2}+1}\ominus a^{L_1}z^{J_1}\dots a^{L_{K-2}-1}.\end{aligned}$$

The last case to deal with is $L_{K-1}=2$ and $J_{K-1}+J_{K-2}\neq 0$ mod s, and again we can conclude thanks to

$$a^{L_1}z^{J_1}\dots z^{J_{K-2}}a^2z^{J_{K-1}}a = (a^{L_1}\dots a^{L_{K-2}}z^{J_{K-2}}a\otimes az^{J_{K-1}}a)\ominus a^{L_1}z^{J_1}\dots a^{L_{K-2}}z^{J_{K-2}+J_{K-1}}a.$$

□

As a corollary, we can get the result also proved in [BV09] (see Theorem 9.3):

Corollary 2.2.2. *Let r_α be an irreducible corepresentation of $C(H_N^{s+})$ with $\alpha = a^{l_1}z^{j_1}\dots a^{l_k}$. Then*

$$\dim(r_\alpha) = \prod_{i=1}^{k} A_{l_i}(\sqrt{N}).$$

Proof. We have $\dim(r_\alpha) = \epsilon_{C(H_N^{s+})}(\chi_\alpha) = \epsilon_{C(S_N^+)}\circ\pi(\chi_\alpha)$ since π is a morphism of Hopf algebras. But the counit on $C(S_N^+)_0$ is given by the evaluation in N. Indeed, an immediate corollary of Theorem 1.1.13 and Proposition 2.1.12, is $\epsilon(\Pi_t) = \Pi_t(N)$ for all polynomials Π_t, which form a basis of $\mathbb{R}[X]$. Now by the previous proposition $\pi(\chi_\alpha)(x) = \prod_{i=1}^k A_{l_i}(\sqrt{x})$, then $\epsilon_{C(S_N^+)}\circ\pi(\chi_\alpha) = \prod_{i=1}^k A_{l_i}(\sqrt{N})$. □

2.3 Haagerup property for quantum reflection groups

In this section we show that duals of the quantum reflection groups $C(H_N^{s+}) = C(\mathbb{Z}_s) *_w C(S_N^+)$, $s \geq 1$ have the Haagerup property for $N \geq 4$.

We still denote by π the canonical surjection $\pi : C(H_N^{s+}) \to C(S_N^+)$ and by $\psi_x = ev_x$ the states on $C(S_N^+)_0 \simeq C([0,N])$ used to show that $C(S_N^+)$ have the Haagerup property (see [Bra12b]). Essentially, we are going to use both morphisms π, ψ_x in this way: we can define states ϕ_x composing these maps, $\psi_x \circ \pi$, where π sends characters of $C(H_N^{s+})$ on characters of $C(S_N^+)$. Thus, we obtain states on the central algebra $C(H_N^{s+})_0$ and, after checking that these states have some decreasing properties, we can use the Theorem 1.2.10 and conclude.

Lemma 2.3.1. *Let ψ_x, $x \in [0,N]$ be the states given by the evaluation in x on the central C^*-algebra $C(S_N^+)_0$. Then for all $x \in [0,N]$, $\phi_x = \psi_x \circ \pi$ is a state on $C(H_N^{s+})_0$.*

Proof. One just has to note that π is Hopf $*$-homomorphism and hence sends $C(H_N^{s+})_0$ to $C(S_N^+)_0$. Then $\psi_x \circ \pi$ is indeed a functional on $C(H_N^{s+})_0$. The rest is clear. □

Notation 2.3.2. *We introduce a proper function on the monoid S (see Theorem 2.1.16). Let L be defined by $L(\alpha) = \sum_{i=1}^{k_\alpha} l_i$ for $\alpha = a^{l_1} z^{j_1} \dots a^{l_{k_\alpha}}$. Notice that for all $R > 0$ the set $B_R = \left\{\alpha = a^{l_1} z^{j_1} \dots a^{l_{k_\alpha}} : L(\alpha) = \sum_{i=1}^{k_\alpha} l_i \leq R\right\} \subset S$ is finite. Thus we get that a net $(f_\alpha)_{\alpha \in S}$ belongs to $c_0(S) \iff \forall \epsilon > 0\ \exists R > 0 : \forall \alpha \in S, (L(\alpha) > R \Rightarrow |f_\alpha| < \epsilon)$. We say that a net $(f_\alpha)_\alpha$ converges to 0 as $\alpha \to \infty$ if $(f_\alpha)_\alpha \in c_0(S)$.*

Proposition 2.3.3. *Let $N \geq 5$ and let χ_α be an irreducible character of $C(H_N^{s+})$ associated to the irreducible corepresentation r_α with $\alpha = a^{l_1} z^{j_1} a^{l_2} \dots a^{l_{k_\alpha}}$. Then for all $x \in [0,N]$*

$$C_\alpha(x) := \frac{\phi_x(\chi_\alpha)}{\dim(r_\alpha)} = \frac{\psi_x \circ \pi(\chi_\alpha)(X)}{\dim(r_\alpha)} = \prod_{i=1}^{k_\alpha} \frac{A_{l_i}(\sqrt{x})}{A_{l_i}(\sqrt{N})}.$$

Moreover $C_\alpha(x)$ converges to 0 as $\alpha \to \infty$ for all $x \in [4, N)$.

Proof. Let $\alpha = a^{l_1} z^{j_1} \dots a^{l_{k_\alpha}}$. We obtain the first assertion using Proposition 2.2.1 and Corollary 2.2.2:

$$\pi(\chi_\alpha)(X) = A_{l_1} \dots A_{l_{k_\alpha}}(\sqrt{X}),$$

$$d_\alpha := \dim(r_\alpha) = \prod_{i=1}^{k_\alpha} A_{l_i}(\sqrt{N}).$$

By Proposition 2.1.7, for any fixed $x \in (4, N)$, there exists a constant $0 < c < 1$ such that $\dfrac{A_l(\sqrt{x})}{A_l(\sqrt{N})} \leq \left(\dfrac{\sqrt{x}}{\sqrt{N}}\right)^{cl}$ for all $l \geq 1$. Then

$$C_\alpha(x) = \frac{\phi_x(\chi_\alpha)}{\dim(r_\alpha)} = \prod_{i=1}^{k_\alpha} \frac{A_{l_i}(\sqrt{x})}{A_{l_i}(\sqrt{N})} \leq \left(\frac{x}{N}\right)^{\frac{c}{2}\sum_i l_i} = \left(\frac{x}{N}\right)^{\frac{c}{2}L(\alpha)} \underset{\alpha\to\infty}{\longrightarrow} 0.$$

□

Proposition 2.3.4. *(Case $N = 4$) Let χ_α be an irreducible character of $C(H_4^{s+})$ associated to the irreducible corepresentation r_α with $\alpha = a^{l_1} z^{j_1} a^{l_2} \dots a^{l_{k_\alpha}}$. Then for all $x \in [0, 4]$*

$$C_\alpha(x) := \frac{\phi_x(\chi_\alpha)}{\dim(r_\alpha)} = \frac{\psi_x \circ \pi(\chi_\alpha)(X)}{\dim(r_\alpha)} = \prod_{i=1}^{k_\alpha} \frac{A_{l_i}(\sqrt{x})}{A_{l_i}(2)}.$$

Moreover $C_\alpha(x)$ converges to 0 as $\alpha \to \infty$ for all $x \in (0, 4)$.

Proof. The proof of the first assertion is similar to the one of the previous proposition. For the second assertion, we use Proposition 2.1.9. We recall that we proved in that proposition that there exists a constant $D < 1$ such that for all $x \in (0, 4)$ and all $l \geq 1$

$$\frac{A_l(\sqrt{x})}{A_l(2)} \leq D. \tag{2.4}$$

Let $\epsilon > 0$ and $x \in (0, 4)$. We want to prove that,

$$\prod_{i=1}^{k_\alpha} \frac{A_{l_i}(\sqrt{x})}{A_{l_i}(2)} < \epsilon \quad \text{for } \alpha \text{ large enough.} \tag{2.5}$$

By (2.4), there exists a $K > 0$ such that $\prod_{i=1}^{k_\alpha} \frac{A_{l_i}(\sqrt{x})}{A_{l_i}(2)} < \epsilon$ for all $\alpha \in S$ with $k_\alpha \geq K$. But by Proposition 2.1.9 there is also an $L > 0$ such that $\dfrac{A_l(\sqrt{x})}{A_l(2)} < \epsilon$ for all $l \geq L$, since this quotient converges to 0.

Now let $\alpha = a^{l_1} z^{j_1} \dots a^{l_{k_\alpha}} \in S$, with $L(\alpha) \geq LK$. Then either $k_\alpha \geq K$, or there exists $i_0 \in \{1, \dots, k_\alpha\}$ such that $l_{i_0} \geq L$. In both case we can get (2.5) since

$$\prod_{i=1}^{k_\alpha} \frac{A_{l_i}(\sqrt{x})}{A_{l_i}(2)} \leq \frac{A_{l_{i_0}}(\sqrt{x})}{A_{l_{i_0}}(2)},$$

each factor of the product being less that one. □

Then we can prove the theorem:

Theorem 2.3.5. *The dual of H_N^{s+} has the Haagerup property for all $N \geq 4$.*

Proof. We follow the proof in [Bra12a] for O_N^+. We prove that the dual of H_N^{s+} has the Haagerup approximation property for all $N \geq 4$ using both previous propositions. Consider the net $(T_{\phi_x})_{x \in I_N}$ with $I_N = (4, N)$ if $N \geq 5$, $I_N = (0, 4)$ if $N = 4$ and

$$T_{\phi_x} = \sum_{\alpha \in Irr(H_N^{s+})} \frac{\phi_x(\chi_{\overline{\alpha}})}{d_\alpha} p_\alpha$$

The ϕ_x are states on $C(H_N^{s+})_0$ so, by Theorem 1.2.10, the T_{ϕ_x} are a unital contractions of $L^2(H_N^{s+})$, and their restrictions to $L^\infty(H_N^{s+})$ are NUCP h-preserving maps. Moreover, Proposition 2.3.4 in the case $N = 4$ and Proposition 2.3.3 in the cases $N \geq 5$, together with the fact that the p_α are finite rank operators, show that for each $x \in I_N$, the operator T_{ϕ_x} is compact. To conclude one has to show that for all $x \in I_N$,

$$||T_{\phi_x} a - a||_{L_2} \underset{x \to N}{\longrightarrow} 0 \tag{2.6}$$

for all $a \in L^\infty(H_N^{s+})$ (via $a \in L^\infty(H_N^{s+}) \hookrightarrow L^2(H_N^{s+})$). First let us prove that it is true for any element $a \in Pol(H_N^{s+})$ i.e. any linear combination of matrix coefficients U_{ij}^α of irreducible corepresentations of $C(H_N^{s+})$ (by linearity, we can do that only for the elements U_{ij}^α). Notice that if $\alpha = a^{l_1} z^{j_1} \ldots z^{j_{k_\alpha - 1}} a^{l_{k_\alpha}}$ then $\overline{\alpha} = a^{l_{k_\alpha}} z^{-j_{k_\alpha - 1}} \ldots z^{-j_1} a^{l_1} = a^{l_{k_\alpha}} z^{s - j_{k_\alpha - 1}} \ldots z^{s - j_1} a^{l_1}$. Thus by Proposition 2.2.1 $\phi_x(\chi_{\overline{\alpha}}) = \psi_x \circ \pi(\chi_{\overline{\alpha}}) = \psi_x \circ \pi(\chi_\alpha) = \phi_x(\chi_\alpha)$. Hence,

$$||T_{\phi_x} U_{ij}^\alpha - U_{ij}^\alpha||_{L^2} = ||U_{ij}^\alpha||_{L_2} \left(1 - \prod_{i=1}^{k_\alpha} \frac{A_{l_i}(\sqrt{x})}{A_{l_i}(\sqrt{N})} \right),$$

so let $x \to N$ and the assertion (2.6) holds for all these matrix coefficient. Now by L^2-density of $Pol(H_N^{s+})$ and the fact that all T_{ϕ_x}, $x \in I_N$, are unital contractions (and thus are uniformly bounded), we obtain that (2.6) is true for any $a \in L^2(H_N^{s+})$. □

Remark 2.3.6. *In [Bic04], it is proved that there is a $*$-algebras isomorphism between $C(H_2^{s+})$ and $C^*(\mathbb{Z}_s * \mathbb{Z}_s \times \mathbb{Z}_2)$ (see Example* 2.5 *and thereafter in that paper). Furthermore, the Haar state on $C^*(\mathbb{Z}_s) *_w C(S_2^+)$ is given by $h = h_1 \otimes h_2$ where h_2 is the Haar state on $C(S_2^+)$ and h_1 is the free product of the Haar states on $C^*(\mathbb{Z}_s)$). Then, it is clear that H_2^{s+} has the Haagerup property by the stability properties of the Haagerup property on groups (see e.g. [CCJ+])*

The algebra $C(H_3^{s+})$ is more complicated and does not reduce to a more comprehensive tensor product as for the case $N = 2$. We are unable at the moment to prove that H_3^{s+} has the Haagerup property.

Chapter 3

The fusions rules for certain free wreath product quantum groups and applications

This Chapter is the text of the chapter [Lem13a]. We find the fusion rules of the free wreath products $\widehat{\Gamma} \wr_* S_N^+$ for any (discrete) group Γ. To do this we describe the spaces of intertwiners between basic corepresentations which allows us to identify the irreducible corepresentations. We then apply the knowledge of the fusion rules to prove, in most cases, several operator algebraic properties of the associated reduced C^*-algebras such as simplicity and uniqueness of the trace. We also prove that the associated von Neumann algebra is a full type II_1-factor and that the dual of $\widehat{\Gamma} \wr_* S_N^+$ has the Haagerup approximation property for all finite groups Γ.

Introduction

Wang constructed in [Wan93] and [Wan95], new examples of compact quantum groups U_N^+ and O_N^+ called the free unitary and free orthogonal quantum groups and introduced also, in [Wan98], the quantum permutation group S_N^+. We recall the definitions of the underlying Woronowicz-C^*-algebras:

- $C(U_N^+) = C^* - \langle u_{ij} :\ 1 \leq, i, j \leq N |\ (u_{ij})_{ij}$ and $(u_{ij}^*)_{ij}$ are unitaries$\rangle$
- $C(O_N^+) = C^* - \langle o_{ij} :\ 1 \leq, i, j \leq N |\ o_{ij}^* = o_{ij}$ and $(o_{ij})_{ij}$ is unitary$\rangle$
- $C(S_N^+) = C^* - \langle v_{ij} :\ 1 \leq, i, j \leq N |\ (v_{ij})_{ij}$ is a magic unitary$\rangle$

which are “free" versions of the commutative C^*-algebras of functions $C(U_N)$, $C(O_N)$, $C(S_N)$. These compact quantum groups were studied by Banica who described, in [Ban97], [Ban96] and [Ban05], their irreducible corepresentations and the fusion rules binding them. This work laid the foundations for the study of the geometric, analytic and combinatoric properties of these quantum groups.

Later, new examples of compact quantum groups appeared. Banica and Speicher introduced the notion of easy quantum groups, [BS09]. They are compact quantum groups whose Woronowicz-C^*-algebras are generated by a unitary matrix (with additional properties). Their intertwiner spaces have a combinatorial description in terms of non-crossing partitions. These compact quantum groups cover the basic examples O_N^+, U_N^+, S_N^+ we mentioned above and include new ones. More recently, Weber [Web13], Raum and Weber [RW12], Freslon and Weber [FW13], investigated these “combinatorial" quantum groups in order to classify them. Earlier in [BV09], Banica and Vergnioux found the fusions rules of the quantum reflection groups H_N^{s+} (for $s \geq 1$ and $N \geq 4$) another family of compact quantum groups introduced in [BBCC11].

In [Bic04], Bichon introduced the notion of free wreath product $A *_w C(S_N^+)$ where A is any unital C^*-algebra (see [Bic04]). Furthermore, Bichon proved that when $\mathbb{G} = (A, \Delta)$ is a compact quantum group, $\mathbb{G} \wr_* S_N^+ = (C(\mathbb{Z}_s) *_w C(S_N^+), \Delta)$ is again a compact quantum group. Bichon, Banica and Vergnioux, proved that H_N^{s+} is the free wreath product of compact quantum groups $\mathbb{Z}_s \wr_* S_N^+$. However, there is no description of the fusion rules of $\mathbb{G} \wr_* S_N^+$, in general except when $\mathbb{G}$ is the dual of $\mathbb{Z}_s$ or $\mathbb{Z}$ corresponding to H_N^{s+}, $s \in [1, \infty]$.

We propose to generalize the description of the fusion rules of $\widehat{\mathbb{Z}_s} \wr_* S_N^+$ to the free wreath products $\widehat{\Gamma} \wr_* S_N^+$ (with the notation above, $A = C^*(\Gamma)$) for *any* (discrete) group Γ. This provides a whole new class of compact quantum groups with an explicit description of the fusion rules.

Another motivation of this work is to pursue the study of the operator algebras associated to compact quantum groups started by Banica ([Ban97]) with the simplicity of $C_r(U_N^+)$. Vergnioux proved in [Ver05] the property of Akemann-Ostrand for $L^\infty(U_N^+)$ and $L^\infty(O_N^+)$ and together with Vaes proved the factoriality, fulness and exactess for $L^\infty(O_N^+)$ in [VV07]. More recently Brannan ([Bra12a], [Bra12b]) proved the Haagerup property for $L^\infty(O_N^+)$, $L^\infty(U_N^+)$ and $L^\infty(S_N^+)$. Freslon proved the weak-amenability of $L^\infty(O_N^+)$, $L^\infty(U_N^+)$ in ([Fre13]) and together with De Commer and Yamashita proved the weak amenability for $L^\infty(S_N^+)$ in [DCFY13]. In each of these results, the knowledge of the fusion rules of the compact quantum groups is a crucial tool to prove the properties of the associated reduced C^* and von Neumann algebras.

From the results proved in this thesis arise the following questions:

- Is it true that if Γ is a discrete group with the Haagerup property, then the dual of $H_N^+(\Gamma) \simeq \widehat{\Gamma} \wr_* S_N^+$ has the Haagerup property ?
- Is it true that the dual of $\widehat{\Gamma} \wr_* S_N^+$ is weakly-amenable ?
- Which other algebraic operator properties possess $C_r(H_N^+(\Gamma))$, $L^\infty(H_N^+(\Gamma))$ (bi-exactness, property RD etc.) ?
- Can one compute the fusion rules of wreaths products $\mathbb{G} \wr_* S_N^+$ if $\mathbb{G}$ is a compact quantum group with known fusion rules ?

The rest of this chapter is organized as follows. The first section is dedicated to recall some definitions and general results on compact quantum groups, the definitions of the quantum permutation group S_N^+ and of the free wreath products by S_N^+, $H_N^+(\Gamma) \simeq \widehat{\Gamma} \wr_* S_N^+$, especially $H_N^{\infty+} \simeq \widehat{\mathbb{Z}} \wr_* S_N^+$.

In the second section, we recall the concept of Tannaka-Krein duality and we describe the intertwiner spaces of certain basic corepresentations in $H_N^+(\Gamma)$ using a canonical arrow from the universal algebra of a certain free product of compact quantum groups onto $C(H_N^+(\Gamma))$. This allows us to compute the fusion rules binding irreducible representations of $H_N^+(\Gamma)$.

In the third section, we propose several applications of this description of the fusion rules:

- The simplicity and the uniqueness of the trace of the reduced C^*-algebra $C_r(H_N^+(\Gamma))$ for all discrete groups $|\Gamma| \geq 4$ and all $N \geq 8$ (in particular $L^\infty(H_N^+(\Gamma))$ is a II_1-factor). We adapt a variant of Powers' methods used by Banica in [Ban97] and we use the simplicity of $C_r(S_N^+)$ for all $N \geq 8$ proved in [Bra12b].
- The fullness of the II_1-factor $L^\infty(H_N^+(\Gamma))$ ($N \geq 8$, any discrete group Γ) which is adapted from the "$14 - \epsilon$ method" which is used in the classical proof of the fact that $L^\infty(F_n)$ has not the property Γ. This application is based upon work by Vaes for the fullness of $L^\infty(U_N^+)$ which can be found in an appendix to [DCFY13].
- We finish this work by extending the main result in [Lem13b] by proving that the dual of the free wreath products $\widehat{\Gamma} \wr_* S_N^+$ has the Haagerup property for all *finite* groups Γ and all $N \geq 4$.

3.1 Preliminaries

In this first section we recall a few facts and results about compact quantum groups and about free wreath products by the quantum permutation groups S_N^+.

A compact quantum group is a pair $\mathbb{G} = (C(\mathbb{G}), \Delta)$ where $C(\mathbb{G})$ is a unital (Woronowicz)-C^*-algebra and $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes_{min} C(\mathbb{G})$ is a unital $*$-homomorphism i.e. they satisfy the coassociativity relation $(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta$, and the cancellation property, that is $span\{\Delta(a)(b \otimes 1) : a, b \in C(\mathbb{G})\}$ and $span\{\Delta(a)(1 \otimes b) : a, b \in C(\mathbb{G})\}$ are norm dense in $C(\mathbb{G}) \otimes C(\mathbb{G})$. These assumptions allow to prove the existence and uniqueness of a Haar state $h : C(\mathbb{G}) \to \mathbb{C}$ satisfying the bi-invariance relations $(h \otimes id) \circ \Delta(\cdot) = h(\cdot)1 = (id \otimes h) \circ \Delta(\cdot)$. In this chapter we will deal with compact quantum groups of Kac type, that is their Haar state h is a trace.

One can consider the GNS representation $\lambda_h : C(\mathbb{G}) \to B(L^2(\mathbb{G}, h))$ associated to the Haar state h of $\mathbb{G} = (C(\mathbb{G}), \Delta)$ and called the left regular representation. We will denote by Λ_h the GNS map. The reduced C^*-algebra associated to $\mathbb{G}$ is then defined by $C_r(\mathbb{G}) = \lambda_h(C(\mathbb{G})) \simeq C(\mathbb{G})/Ker(\lambda_h)$ and the von Neumann algebra by $L^\infty(\mathbb{G}) = C_r(\mathbb{G})''$. One can prove that $C_r(\mathbb{G})$ is again a Woronowicz-C^*-algebra, that $h \circ \lambda_h$ is its Haar state and that it extends to $L^\infty(\mathbb{G})$. We will denote simply by Δ and h the coproduct and Haar state on $C_r(\mathbb{G})$. The full version (also called universal) of a compact quantum group $\mathbb{G}$, will be denoted by $\mathbb{G}_u = (C_u(\mathbb{G}), \Delta)$.

An N-dimensional (unitary) representation $u = (u_{ij})_{ij}$ of $\mathbb{G}$ (or corepresentation of $C(\mathbb{G})$) is a (unitary) matrix $u \in M_N(C(\mathbb{G})) \simeq C(\mathbb{G}) \otimes B(\mathbb{C}^N)$ such that for all $i, j \in \{1, \dots, N\}$, one has

$$\Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj}.$$

The matrix $\overline{u} = (u_{ij}^*)$ is called the conjugate of $u \in M_N(C(\mathbb{G}))$ and in general it is not necessarily unitary even if u is. However all the compact quantum groups we will deal with are of Kac type and in this case the conjugate of a unitary corepresentation is also unitary.

An intertwiner between two corepresentations

$$u \in M_{N_u}(C(\mathbb{G})) \text{ and } v \in M_{N_v}(C(\mathbb{G}))$$

is a matrix $T \in M_{N_u, N_v}(\mathbb{C})$ such that $v(1 \otimes T) = (1 \otimes T)u$. We say that u is equivalent to v, and we note $u \sim v$, if there exists an invertible intertwiner between u and v. We denote by $Hom_{\mathbb{G}}(u, v)$ the space of intertwiners between u and v. A corepresentation u is said to be irreducible if $Hom_{\mathbb{G}}(u, u) = \mathbb{C}id$. We denote by $Irr(\mathbb{G})$ the set of equivalence classes of irreducible representations of $\mathbb{G}$.

We recall that $C(\mathbb{G})$ contains a dense $*$-subalgebra denoted by $Pol(\mathbb{G})$ and linearly generated by the coefficients of the irreducible representations of $\mathbb{G}$. The coefficients of a $\mathbb{G}$-representation r are given by $(id \otimes \phi)(r)$ for some $\phi \in B(H_r)^*$ if the corepresentation acts on the Hilbert space H_r. This algebra has a Hopf-$*$-algebra structure and in particular there is a $*$-antiautomorphism $\kappa : Pol(\mathbb{G}) \to Pol(\mathbb{G})$ which acts on the coefficients of an irreducible corepresentation $r = (r_{ij})$ as follows $\kappa(r_{ij}) = r^*_{ij}$. This algebra is also dense in $L^2(\mathbb{G}, h)$. Since h is faithful on the $*$-algebra $Pol(\mathbb{G})$, one can identify $Pol(\mathbb{G})$ with its image in the GNS-representation $\lambda_h(C(\mathbb{G}))$. We will denote by χ_r the character of the irreducible corepresentation $r \in Irr(\mathbb{G})$, that is $\chi_r = (id \otimes Tr)(r)$.

In the Kac type case, the right regular representation $\rho_h : C(\mathbb{G}) \to B(L^2(\mathbb{G}, h))$ is given by $\rho_h(x)\Lambda_h(y) = \Lambda_h(y\kappa(x))$. It commutes with λ_h and if $\mathbb{G}$ is full, one can consider the adjoint representation $(\lambda_h, \rho_h) \circ \Delta : C_u(\mathbb{G}) \to B(L^2(\mathbb{G}_u, h))$. This representation acts on the irreducible characters as follows

$$ad(\chi_r)(z) = \sum_{ij} r_{ij} z r^*_{ij}.$$

Notice that the map $z \mapsto ad(\chi_r)(z)$ is completely positive for all $r \in Irr(H^+_N(\Gamma))$.

In [Ban97], the author uses the notion of support of an element $x \in Pol(\mathbb{G})$. The support of $x \in Pol(\mathbb{G})$ is denoted $supp(x)$ and defined as the smallest subset $G \subset Irr(\mathbb{G})$ such that x is a linear combination of certain coefficients of elements $r \in G$. In other words,

$$r \notin supp(x) \Leftrightarrow h(xr^*_{ij}) = 0, \text{ for all coefficients } r_{ij} \text{ of } r.$$

A fundamental and basic family of examples of compact quantum groups are the quantum reflection groups S^+_N, see Definition 1.1.12. Recall that $C(S^+_N)$ is generated N^2 elements v_{ij} such that the matrix $v = (v_{ij})$ is a magic unitary.

In the cases $N = 2, 3$, one obtains the usual algebras $C(\mathbb{Z}_2), C(S_3)$ since a magic unitary of size 2 (respectively 3) is composed of commuting projections as one can see using the Fourier transformation over $\mathbb{Z}_2$, resp. $\mathbb{Z}_3$. If $N \geq 4$, one can find an infinite dimensional quotient of $C(S^+_N)$ so that $C(S^+_N)$ is not isomorphic to $C(S_N)$, see e.g. [Wan98], [Ban05].

The representation theory of S^+_N is well known and recalled in Theorem 1.1.13.

In [Wan95], Wang defined the free product $\mathbb{G} = \mathbb{G}_1 * \mathbb{G}_2$ of compact quantum groups, showed that $\mathbb{G}$ is still a compact quantum group and gave a description of the irreducible representations of $\mathbb{G}$ as alternating tensor products of nontrivial irreducible representations. Indeed, one can identify the set $Irr(\mathbb{G})$ of irreducible representation of $\mathbb{G} = \mathbb{G}_1 * \mathbb{G}_2$ with the set of alternating words $Irr(\mathbb{G}_1) * Irr(\mathbb{G}_2)$ and the fusion rules may be recursively described as follows:

- if the words $x, y \in Irr(\mathbb{G})$ end and start respectively in $Irr(\mathbb{G}_i)$ and $Irr(\mathbb{G}_j)$ with $j \neq i$ then $x \otimes y$ is an irreducible representation of $\mathbb{G}$ corresponding to the concatenation $xy \in Irr(\mathbb{G})$.

- if $x = vz$ and $y = z'v$ with $z, z' \in Irr(\mathbb{G}_i)$ then

$$x \otimes y = \bigoplus_{1 \neq t \subset z \otimes z'} vtw \oplus \delta_{\overline{z}, z'}(v \otimes w)$$

  where the sum runs over all non-trivial irreducible corepresentations $t \in Irr(\mathbb{G}_i)$ contained in $z \otimes z'$.

We will use this fact to describe the fusion rules of another type of product of compact quantum groups: the free wreath products $\widehat{\Gamma} \wr_* S_N^+$, with Γ a discrete group (see section 3.2).

We refer to Definition 1.1.14 and Theorem 1.1.15 for the definition and a fundamental result on free wreath products.

The following examples are fundamental for the rest of this chapter.

Example 3.1.1. *([Bic04, Example 2.5]) Let Γ be a (discrete) group, $N \geq 2$. Let $A_N(\Gamma)$ be the universal C^*-algebra with generators $a_{ij}(g), 1 \leq i, j \leq N, g \in \Gamma$ together with the following relations:*

$$a_{ij}(g)a_{ik}(h) = \delta_{jk}a_{ij}(gh) \quad ; \quad a_{ji}(g)a_{ki}(h) = \delta_{jk}a_{ji}(gh) \tag{3.1}$$

$$\sum_{l=1}^{N} a_{il}(e) = 1 = \sum_{l=1}^{N} a_{li}(e), \tag{3.2}$$

and involution $a_{ij}(g)^* = a_{ij}(g^{-1})$. Then $H_N^+(\Gamma) := (A_N(\Gamma), \Delta)$ is a compact quantum group with:

$$\Delta(a_{ij}(g)) = \sum_{k=1}^{N} a_{ik}(g) \otimes a_{kj}(g). \tag{3.3}$$

We have for all $g \in \Gamma$, $\epsilon(a_{ij}(g)) = \delta_{ij}$ and $S(a_{ij}(g)) = a_{ji}(g^{-1})$. Furthermore, $H_N^+(\Gamma)$ is isomorphic, as compact quantum groups, with $\widehat{\Gamma} \wr_* S_N^+$. Consider the following important special cases:

1. *If $\Gamma = \mathbb{Z}_s$ for an integer $s \geq 1$, one gets the quantum reflection groups H_N^{s+} (see [BBCC11] and [BV09]). $C(H_N^{s+})$ is the universal C^*-algebra generated by N^2 normal elements U_{ij} such that for all $1 \leq i, j \leq N$:*

   (a) $U = (U_{ij})$ and ${}^tU = (U_{ji})$ are unitary,

   (b) $U_{ij}U_{ij}^*$ *is a projection,*

   (c) $U_{ij}^s = U_{ij}U_{ij}^*$,

   (d) $\Delta(U_{ij}) = \sum_{k=1}^N U_{ik} \otimes U_{kj}$.

2. *If* $\Gamma = \mathbb{Z}$, *one gets* $H_N^{\infty+} = (C(H_N^{\infty+}), \Delta)$ *where* $C(H_N^{\infty+})$ *and* Δ *are defined as above except that one removes the relations (1c) above.*

We will use the following proposition to find the fusion rules of $H_N^+(\Gamma)$.

Proposition 3.1.2. *Let* Γ *be a finitely generated group* $\Gamma = \langle \gamma_1, \dots, \gamma_p \rangle$. *We have canonical arrows*

$$C(H_N^{\infty+})^{*p} \xrightarrow{\pi_1} C(H_N^+(\Gamma)) \xrightarrow{\pi_2} C(S_N^+)$$

given, for all $1 \leq i_t, j_q \leq N$, *by*

$$\pi_1\left(U_{i_1j_1}^{(r_1)} \dots U_{i_kj_k}^{(r_k)}\right) = a_{i_1j_1}(\gamma_{r_1}) \dots a_{i_kj_k}(\gamma_{r_k}),$$

and for all $1 \leq i, j \leq N, g \in \Gamma$ *by*

$$\pi_2(a_{ij}(g)) = v_{ij}.$$

where $U_{ij}^{(r)}$ *is the coefficient of the fundamental corepresentation of* $C(H_N^{\infty+})$ *chosen in the* r*-th copy of* $C(H_N^{+\infty})$ *in the free product* $C(H_N^{\infty+})^{*p}$ *with* $1 \leq r \leq p$.

Proof. The existence of the arrow π_2 is clear by universality of $C(H_N^+(\Gamma))$. For π_1, notice that for all $1 \leq r \leq p$ there is, by universality of $C(H_N^{\infty+})$, an arrow $\pi_1^{(r)}$ such that $\pi_1^{(r)}(U_{ij}^{(r)}) = a_{ij}(\gamma_r)$ for all $1 \leq i, j \leq N$. Then $\pi_1 = \star_{r=1}^p \pi_1^{(r)} : C(H_N^{\infty+})^{*p} \to C(H_N^+(\Gamma))$ satisfies for all $1 \leq r_1, \dots, r_k \leq p$,

$$\pi_1\left(U_{ij}^{(r_1)} \dots U_{ij}^{(r_k)}\right) = \pi_1^{(r_1)}\left(U_{ij}^{(r_1)}\right) \dots \pi_1^{(r_k)}\left(U_{ij}^{(r_k)}\right) = a_{i_1j_1}(\gamma_{r_1}) \dots a_{i_kj_k}(\gamma_{r_k}).$$

□

We refer to Theorem 1.1.18 for the description of the corepresentations and fusion rules proved in [BV09] for the compact quantum reflection groups $H_N^{s+}, H_N^{\infty+}$ $(N \geq 4)$.

We want to prove that one can generalize this description of the irreducible corepresentations and fusion rules to $H_N^+(\Gamma) = \widehat{\Gamma} \wr_* S_N^+$ for any (discrete) group Γ, $N \geq 4$, and get several applications that one can deduce from the knowledge of the fusion rules.

We recall that it is already known in the case $\Gamma = \mathbb{Z}_s$ ($s \geq 1$ finite):

Theorem 3.1.3. *([Lem13b]) The dual of* H_N^{s+} *has the Haagerup property for all* $N \geq 4, s \in [1, +\infty)$.

3.2 Fusion rules for some free wreath products by the quantum permutation group

In this section Γ is any (discrete) group Γ, N is an integer $N \geq 4$. We are going to describe the irreducible representations of $H_N^+(\Gamma) = \widehat{\Gamma} \wr_* S_N^+$ and the fusion rules binding them. To fulfill this, we are going to use techniques introduced in [Ban96], [Ban97], [Ban99b] and developed in [BBCC11] and [BV09].

More precisely, let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group such that $C(\mathbb{G})$ is generated by the coefficients of a certain family $(v_i)_{i\in I}$ of finite dimensional $\mathbb{G}$-representations. Denote by $Rep(\mathbb{G})$ the complete C^*-tensor category with conjugates of all finite dimensional representations of $\mathbb{G}$ (see e.g. [Wor88] and [NT] for the definitions of such rigid monoidal categories). We will keep the following notation in the sequel of this chapter:

Notation 3.2.1. *We denote by $Tens(\mathbb{G}, (v_i)_{i\in I}) \subset Rep(\mathbb{G})$ the full tensor subcategory with:*

- *objects: all the tensor products between corepresentations v_i and $\overline{v_i}$,*
- *morphisms: intertwiners between such tensor products.*

$Tens(\mathbb{G}, (v_i)_{i\in I})$ is contained in the category of (all) linear maps between tensor products of the representation spaces H_i of the corepresentations v_i. Denote this category $Vect(H_i)$.

We will denote by v the generating matrix of S_N^+ and denote by $a_{kl}(\gamma_i)$ the generating elements of $H_N^+(\Gamma)$. Notice that, if $U^{(i)}$ denotes the i-th copy of the fundamental representation of $H_N^{\infty+}$ in $(H_N^{\infty+})^{*p}$, then the arrows of Proposition 3.1.2 at the level of the objects

$$U^{(i)} \mapsto (a_{kl}(\gamma_i))_{k,l} \mapsto v,$$

give functors at the level of the categories

$$Tens\left((H_N^{\infty+})^{*p}, \left\{U^{(i)}\right\}_{i=1}^{p}\right) \to Tens\left(H_N^+(\Gamma), a(\gamma_i) : i = 1, \dots, p\right) \to Tens(S_N^+, v). \tag{3.4}$$

It is known (and recalled in the next section) that the tensor categories $Tens(H_N^{s+}, U)$ and $Tens(S_N^+, v)$ have a diagrammatic description in terms of non-crossing partitions: morphisms (i.e. intertwiners between representations) can be described by (certain) non-crossing partitions. We use this fact to obtain a diagrammatic description of the tensor category $Tens(H_N^+(\Gamma), a(\gamma_i) : i = 1, \dots, p)$. The inclusions (3.4) above, together with the diagrammatic description of $Tens\left((H_N^{\infty+})^{*p}, U^{(i)}\right)$ (which we will obtain in the subsection 3.2.2), will allow us to conclude.

Before investigating these questions, we recall that the fusion rules of $H_N^+(\Gamma)$ are known in the case $N = 2$, see [Bic04]. In the sequel, we assume that $N \geq 4$. The case $N = 3$ is not investigated in this thesis.

3.2.1 Non-crossing partitions, diagrams. Tannaka-Krein duality

The following paragraph recalls a few notions on non-crossing partitions, see e.g. [BV09] for more informations. We repeat certain definitions and results of the Chapter 1 since we will use certain less standard notation.

Definition 3.2.2. *We denote by $NC(k,l)$ the set of non-crossing diagrams between k upper points and l lower points, that is the non-crossing partitions of the sets with $k+l$ ordered elements, with the following pictorial representation:*

$$\left\{ \begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \mathscr{P} & & \\ \cdot & \cdot & \cdot & \cdot & \end{array} \right\}$$

with k-upper points, l-lower points and $\mathscr{P}$ is a a diagram composed of strings which connect certain upper and/or lower points and which do not cross one another.

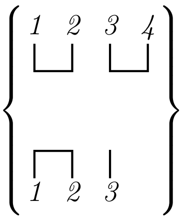
Such non-crossing partitions give rise to new ones by tensor product, composition and involution:

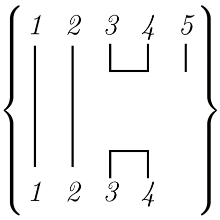
Definition 3.2.3. *Let $p \in NC(k,l)$ and $q \in NC(l,m)$. Then, the tensor product, composition and involution of the partitions p,q are obtained by horizontal concatenation, vertical concatenation and upside-down turning:*

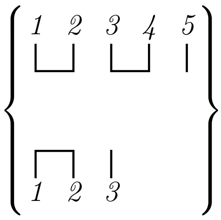
$$p \otimes q = \{\mathscr{P}\mathscr{Q}\}, \ pq = \left\{ \begin{array}{c} \mathscr{Q} \\ \mathscr{P} \end{array} \right\} - \{closed\ blocks\}, \ p^* = \{\mathscr{P}^{\downarrow}\}.$$

The composition pq is only defined if the number of lower points of q is equal to the number of upper points of q. When one identifies the lower points of p with the upper points of q, closed blocks might appear, that is strings which are connected neither to the new upper points nor to the new lower points. These blocks are discarded from the final pictorial representation.

Example 3.2.4. *Following the rules stated above (discarding closed blocks and following the lines when one identifies the upper points of p with the lower points of q), we get*

if $p =$ 

and $q =$ 

then $pq =$ 

.

From non-crossing partitions $p \in NC(k,l)$ naturally arise linear maps $T_p : \mathbb{C}^{N^{\otimes k}} \to \mathbb{C}^{N^{\otimes l}}$:

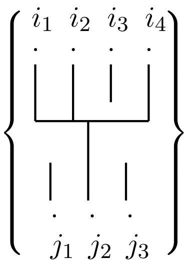
Definition 3.2.5. *Consider (e_i) the canonical basis of $\mathbb{C}^N$. Associated to any non-crossing partition $p \in NC(k,l)$ is the linear map $T_p \in B\left(\mathbb{C}^{N^{\otimes k}}, \mathbb{C}^{N^{\otimes l}}\right)$:*

$$T_p(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_{j_1,\dots,j_l} \delta_p(i,j) e_{j_1} \otimes \cdots \otimes e_{j_l}$$

where i (respectively j) is the k-tuple $(i_1,\dots,i_k)$ (respectively l-tuple $(j_1,\dots,j_l)$) and $\delta_p(i,j)$ is equal to:

1. 1 *if all the strings of p join equal indices,*
2. 0 *otherwise.*

Example 3.2.6. *We consider an element $p \in NC(4,3)$, choose any tuples $i = (i_1, i_2, i_3, i_4)$ and $j = (j_1, j_2, j_3)$, and put them on the diagram:*

$p =$ 

Then $\delta_p(i,j) = \begin{cases} 1 & \text{if } i_1 = i_2 = i_4 = j_2 \\ 0 & \text{otherwise.} \end{cases}$

Example 3.2.7. *We give basic examples of such linear maps*

(i.) $T\_\left\{\,|\,\right\} = id_{\mathbb{C}^N}$

(ii.) $T\_\{\bigcap\}(1) = \sum_a e_a \otimes e_a$

Tensor products, compositions and involutions of diagrams behave as follows with respect to the associated linear maps:

Proposition 3.2.8. *([BS09, Proposition 1.9] Let p, q be non-crossing partitions and $b(p, q)$ be the number of closed blocks when performing the vertical concatenation (when it is defined). Then:*

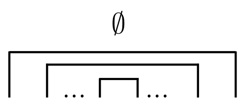
1. $T_{p\otimes q} = T_p \otimes T_q$,
2. $T_{pq} = n^{-b(p,q)} T_p T_q$,
3. $T_{p^*} = T_p^*$.

We will keep the following notation in the sequel:

Notation 3.2.9.

- *We will denote by NC the collection of all the sets $NC(k, l)$ which form a monoidal category with involution and with $\mathbb{N}$ as a set of objects.*
- *We will denote by NC^I the set of non-crossing partitions whose objects are tuples of elements of the set I and morphisms between a k-tuple and an l-tuple are the ones of $NC(k, l)$.*
- *An important example (see Theorem 3.2.12) is $NC^{\mathbb{Z}_s}$ which contains the subcategory NC_s defined as follows: objects are the same as the ones of $NC^{\mathbb{Z}_s}$ and morphisms $p \in NC^{\mathbb{Z}_s}(\underline{i}, \underline{j})$ are non-crossing partitions decorated by a k-tuple $\underline{i} = (i_1, \dots, i_k) \in \mathbb{Z}_s^k$ and an l-tuple $\underline{j} = (j_1, \dots, j_l) \in \mathbb{Z}_s^l$ having the property that, putting $\underline{i}$ on the upper row of p and $\underline{j}$ on the lower row of p, then in each block, the sum of i indices equals the sum of j indices modulo s. In the case $s = \infty$, we make the convention that this equality modulo s is equality.*

The Proposition 3.2.8 implies easily that the collection of spaces $span\{T_p : p \in NC(k, l)\}$ form a C^*-tensor category with $\mathbb{N}$ as a set of objects. Furthermore, this tensor category has conjugates since the partitions of type

$r = \Big\{$  $\Big\} \in NC(\emptyset; 2k)$

are non-crossing and since the following conjugate equations hold:

$$(T_r^* \otimes id) \circ (id \otimes T_r) = id = (id \otimes T_r^*) \circ (T_r \otimes id). \tag{3.5}$$

Similar arguments show that the collection of spaces $span\{T_p : p \in NC_s(\underline{i}, \underline{j})\}$ form a C^*-tensor category with conjugates.

In addition to notation 3.2.1 and 3.2.9, we will use the following one:

Notation 3.2.10. *We will denote by Lin the "projective" functor from NC^I to $Vect(H_i)$, where the H_i are copies of $\mathbb{C}^N$ for some fixed N, defined as follows:*

- $Lin(i_1, \dots, i_k) = H_{i_1} \otimes \dots \otimes H_{i_k} : (i_1, \dots, i_k) \in I^k$,
- $Lin(p) = T_p \in B\left(\mathbb{C}^{N^{\otimes k}}, \mathbb{C}^{N^{\otimes l}}\right) : p \in NC^I$.

Remark 3.2.11. *The terms "projective" above is used because of the numerical factor appearing in the formula $T_{pq} = n^{-b(p,q)} T_p T_q$ above (Proposition 3.2.8). Lin is then a functor when one replaces the target vector spaces by the associated projective spaces where one quotients by the colinearity equivalence relation.*

Using notation 3.2.1, 3.2.10 and 3.2.9 we can now give a homogeneous result concerning the diagrammatic description of $Tens(S_N^+, v)$ and $Tens(H_N^{s+}, U)$ (see [Ban99b], [BV09]) that we will generalize in the next subsections. All the categories considered below are then contained in $Vect(H_i)$ as defined above:

Theorem 3.2.12. *Let $N \geq 2$, $s \in [1, \infty]$, v be the fundamental corepresentation of $C(S_N^+)$ and U be the fundamental corepresentation of $C(H_N^{s+})$ (see section 3.1).*

1. *$Tens(S_N^+, v) = span\{Lin(NC)\}$ i.e. for all $k, l \in \mathbb{N}$*
$$Hom(v^{\otimes k}, v^{\otimes l}) = span\{T_p : p \in NC(k, l)\}.$$
2. *$Tens(H_N^{s+}, \{U_i\}) = span\{Lin(NC_s)\}$ i.e. for all $k, l \in \mathbb{N}$ and $i_t, j_q \in \mathbb{Z}_s$ (the case $s = \infty$ corresponds to $\mathbb{Z}$)*
$$Hom(U_{i_1} \otimes \dots \otimes U_{i_k}, U_{j_1} \otimes \dots \otimes U_{j_l}) = span\{T_p : p \in NC_s(\underline{i}; \underline{j})\}.$$
3. *Moreover, the linear maps $T_p, p \in NC(k, l)$ are linearly independent for all $N \geq 4$.*

We recall that in a C^*-tensor category with conjugates, we have the following Frobenius reciprocity theorem (see [Wor88] and [NT]) that we will use in Proposition 3.2.15.

Theorem 3.2.13. *Let $\mathscr{C}$ be a C^*-tensor category with conjugates. If an object $U \in \mathscr{C}$ has a conjugate, with R and $\overline{R}$ solving the conjugate equations (see [NT, Definition 2.2.1], or (3.5) above), then the map*
$$Mor(U \otimes V, W) \to Mor(V, \overline{U} \otimes W), T \mapsto (id_{\overline{U}} \otimes T)(R \otimes id_V)$$
is a linear isomorphism with inverse $S \mapsto (\overline{R}^* \otimes id_W)(id_U \otimes S)$.

The next proposition is an application of Woronowicz's Tannaka-Krein duality and will be useful when computing the tensor category of certain compact quantum groups (see Theorem 3.2.20 below and compare with [BBCC11, Theorem 12.1]).

Proposition 3.2.14. *Let $\mathbb{G}_1 = (C(\mathbb{G}_1), \{u_i\})$ and $\mathbb{G}_2 = (C(\mathbb{G}_2), \{v_j\})$ be two compact quantum groups such that $C(\mathbb{G}_1), C(\mathbb{G}_2)$ are generated by the coefficients of some corepresentations $\{u_i\}, \{v_j\}$.*

Suppose that there is a surjective morphism $\pi : C(\mathbb{G}_1) \to C(\mathbb{G}_2)$ intertwining the coproducts (i.e. $\widehat{\mathbb{G}_2} \subset \widehat{\mathbb{G}_1}$). Suppose furthermore that $\ker(\pi)$ is generated by intertwining relations that is by a set $\mathscr{R}$ of linear maps T which are morphisms in $Vect(H_{u_i})$ giving equations in $C(\mathbb{G}_1)$. Then $Tens(\mathbb{G}_2, v_j)$ is generated as a rigid monoidal C^*-tensor category by $Tens(\mathbb{G}_1)$ and $\mathscr{R}$:

$$Tens(\mathbb{G}_2, \pi(u_i)) = \langle Tens(\mathbb{G}_1, u_i), \mathscr{R}\rangle.$$

Proof. Let $\mathbb{G}_2'$ be the compact quantum group, obtained by Tannaka-Krein duality, whose representation category is the completion of $\langle \pi(Tens(\mathbb{G}_1, u_i)), \mathscr{R}\rangle$. By construction, the sets of the intertwining relations in $\mathbb{G}_2'$ and $\mathbb{G}_2$ coincide: they are composed of the relations in $\mathbb{G}_1$ and the additional ones described by $\mathscr{R}$. Thus, the morphism $\pi : C(\mathbb{G}_1) \to C(\mathbb{G}_2)$, with kernel $\mathscr{R}$, factorizes into an isomorphism of compact quantum groups $\pi' : C(\mathbb{G}_2') = C(\mathbb{G}_1)/\mathscr{R} \to C(\mathbb{G}_2)$. □

3.2.2 Free product of compact matrix quantum groups and intertwiner spaces

We are going to find a diagrammatic description of the spaces of intertwiners of the free product $(H_N^{\infty+})^{*p}$. We prove a more general result on the free product of a finite family of compact matrix quantum groups $\mathbb{G} = \star_{i=1}^p \mathbb{G}_i$. We will use the following notation: if U_i is the fundamental corepresentation matrix of the compact quantum group $\mathbb{G}_i$ then U_i^ϵ will denote U_i if $\epsilon = 1$ and its conjugate $\overline{U_i}$ when $\epsilon = -1$.

Proposition 3.2.15. *Let $\mathbb{G}$ be a free product of a finite family of compact matrix quantum groups: $\mathbb{G} = \star_{i=1}^p (\mathbb{G}_i, U_i)$. We denote by $\mathscr{T}$ the category generated by the categories $Tens(\mathbb{G}_i, U_i)$ that is*

- *objects are tensor products $U_{r_1}^{\epsilon_1} \otimes \cdots \otimes U_{r_k}^{\epsilon_k}$ ($r_i \in \{1, \ldots, p\}$),*
- *morphisms are linear combinations and compositions of morphisms of the type $id \otimes g \otimes id$ where g is a morphism in a certain category $Tens(\mathbb{G}_i, U_i)$.*

Then we have $Tens(\mathbb{G}, \{U_i\}) = \mathscr{T}$.

Proof. We first claim that $\mathscr{T} \subset Tens(\mathbb{G}, \{U_i\})$. Indeed, if

$$g \in Hom_{\mathbb{G}_r}(U_r^{\epsilon_1} \otimes \cdots \otimes U_r^{\epsilon_k}, U_r^{\eta_1} \otimes \cdots \otimes U_r^{\eta_l})$$

(for a fixed $r \in \{1, \ldots, p\}$), then we clearly also have

$$g \in Hom_{\mathbb{G}}(U_r^{\epsilon_1} \otimes \cdots \otimes U_r^{\epsilon_k}, U_r^{\eta_1} \otimes \cdots \otimes U_r^{\eta_l})$$

and moreover $Tens(\mathbb{G}, \{U_i\})$ is stable under the operations used to generate $\mathscr{T}$.

Now, we prove the other inclusion $Tens(\mathbb{G}, \{U_i\}) \subset \mathscr{T}$. The Frobenius reciprocity allows us to restrict to the cases $k = 0$: indeed the duality maps T_r (see (3.5)) are compositions of maps $id \otimes T_{r_i} \otimes id$ which are in $\mathscr{T}$. We have to prove that the morphisms $\mathbb{C} \to U_{s_1}^{\eta_1} \otimes \cdots \otimes U_{s_l}^{\eta_l}$ (i.e. the fixed vectors of this latter tensor product) are linear combinations of compositions of maps of the type $id \otimes g \otimes id$, where g is a fixed vector for some factor G_i.

Let T be a morphism $T : \mathbb{C} \to U_{s_1}^{\eta_1} \otimes \cdots \otimes U_{s_l}^{\eta_l}$ and set $V = U_{s_1}^{\eta_1} \otimes \cdots \otimes U_{s_l}^{\eta_l}$. We will call any tensor product

$$U_s^{\eta_t} \otimes \cdots \otimes U_s^{\eta_{t+m}} = \bigotimes_{r=t}^{t+m} U_s^{\eta_r},$$

a sub-block of V if this is a "sub-tensor word" of V such that $s_t = s_{t+1} = \cdots = s_{t+m} = s$ ($1 \leq t \leq \cdots \leq t + m \leq l$) i.e. a tensor word in U_s and $\overline{U_s}$ coming from the same copy $\mathbb{G}_i$; such a sub-block will be called *maximal* when $s_{t-1} \neq s \neq s_{t+m+1}$ (whenever this is well defined).

We are going to prove the desired assertion by induction over the number of maximal sub-blocks. The initialization corresponds to the case where there is only one maximal sub-block i.e. in this case each copy $U_{s_i}^{\eta_i}$ comes from the same factor $\mathbb{G}_{s_i}$ and thus the assertion is clear.

We denote the maximal sub-blocks by

$$B_i = \bigotimes_{r=t_i}^{t_i+m_i} U_{s_i}^{\eta_r}, i = 1, \ldots, k$$

($k \geq 1$) with constant index s_i and we set for all $i = 1, \ldots, k$

$$C_i = \bigotimes_{\substack{r<t_i, \\ r>t_i+m_i}} U_{s_r}^{\eta_r}.$$

We fix $(S_i^j)_j$ an orthonormal basis of $Fix_{B_i} := Hom_{\mathbb{G}_i}(\mathbb{C}, B_i)$, note that $S_i^j \in \mathscr{T}$.

Let $\forall i = 1, \dots, k$, $P_i : B_i \to B_i$ be the orthogonal projections from the space of B_i to Fix_{B_i}, that is, with our notation $P_i = \sum_j S_i^j S_i^{j*}$. Notice that

$$(id \otimes P_i \otimes id) \circ T = \sum_j (id \otimes S_i^j \otimes id) \circ (id \otimes S_i^{j*} \otimes id) \circ T,$$

and that for all j

$$[(id \otimes S_i^{j*} \otimes id) \circ T : \mathbb{C} \to C_i] \in Hom_{\mathscr{T}}(\mathbb{C}, C_i)$$

since C_i has less maximal sub-blocks than V. So, we obtain that

$$[(id \otimes P_i \otimes id) \circ T : \mathbb{C} \to V] = \sum_j \lambda_j (id \otimes g_{j,1} \otimes id) \circ \cdots \circ g_{j,r} \tag{3.6}$$

is a linear combination of composition of maps of type $id \otimes g_j \otimes id$ where g_j is a fixed vector for some factor $\mathbb{G}_j$, in other words $(id \otimes P_i \otimes id) \circ T$ is a morphism in $\mathscr{T}$.

On the other hand, we can write

$$T = (P_1 + (id - P_1)) \otimes \cdots \otimes (P_i + (id - P_i)) \otimes \cdots \otimes (P_k + (id - P_k)) \circ T. \tag{3.7}$$

We denote $\forall i = 1, \dots, k$,

$$P_i^{(\epsilon_i)} = \begin{cases} P_i & \text{if } \epsilon_i = 1, \\ id - P_i & \text{if } \epsilon_i = -1. \end{cases}$$

We are going to expand (3.7) and conclude. The properties of the fusion rules of a free product of compact quantum groups recalled in Section 3.1 yield that $P_1^{(-1)} \otimes \cdots \otimes P_k^{(-1)}$ maps V onto some direct sum

$$\bigoplus_{r_i \neq 1} r_1^{(s_1)} \otimes \cdots \otimes r_k^{(s_k)}.$$

Indeed, each projection $P_i^{(-1)}$ maps the space of B_i on the orthogonal complement of Fix_{B_i} so that $P_i^{(-1)}(B_i)$ decomposes as a direct sum of *non-trivial* irreducible $\mathbb{G}_i$-representations. Hence, since the sub-blocks are maximal, the calculation rules of the alternated tensor words of corepresentations in a free product of compact quantum groups (see section 3.1) yield that the tensor product $P_1^{(-1)}(B_1) \otimes \cdots \otimes P_k^{(-1)}(B_k)$ decomposes into non-trivial irreducible representations of the free product quantum group.

Now, since $Im(T)$ is a copy of the trivial corepresentation $1 \subset V$, we have $P_1^{(-1)} \otimes \cdots \otimes P_k^{(-1)} \circ T = 0$. Then we get

$$T = \sum_{(\epsilon_1,\ldots,\epsilon_k)\in\Omega} \left(P_1^{(\epsilon_1)} \otimes \cdots \otimes P_k^{(\epsilon_k)}\right) \circ T,$$

where $\Omega = \{-1,+1\}^k \backslash \{(-1,-1,\ldots,-1)\}$.

Hence we can write

$$\begin{aligned} T &= \sum_{(\epsilon_1,\ldots,\epsilon_k)\in\Omega} \left(P_1^{(\epsilon_1)} \otimes \cdots \otimes (P_{i_0}) \otimes \cdots \otimes P_k^{(\epsilon_k)}\right) \circ T \qquad (\epsilon_{i_0} = 1) \\ &= \sum_{(\epsilon_1,\ldots,\epsilon_k)\in\Omega} \left(P_1^{(\epsilon_1)} \otimes \cdots \otimes (id) \otimes \cdots \otimes P_k^{(\epsilon_k)}\right) \circ (id \otimes P_{i_0} \otimes id) \circ T. \end{aligned}$$

and T is as announced by (3.6) and the definition of the maps P_i.

□

3.2.3 Intertwinner spaces in $H_N^+(\Gamma)$

Before proving a similar result for $H_N^+(\Gamma) = \widehat{\Gamma} \wr_* S_N^+$ as Theorem 1.1.18, we give a collection of corepresentations, called basic corepresentations, for $H_N^+(\Gamma) = (C(H_N^+(\Gamma)), \Delta)$ and then describe their intertwiner spaces in terms of diagrams. With the notation of Example 3.1.1, we can obtain the following result.

Proposition 3.2.16. *The algebra $C(H_N^+(\Gamma))$ has a family of N-dimensional (basic) unitary corepresentations $\{a(g) : g \in \Gamma\}$ satisfying the conditions:*

1. *for all $g \in \Gamma$, $a(g)_{ij} = a_{ij}(g)$*
2. *$\overline{a(g)} = a(g^{-1})$.*

Proof. This is clear, with the relations (3.1), (3.2), (3.3) , that setting $a(g) = (a_{ij}(g))_{i,j\in\{1,\ldots,N\}}$ gives the desired family of corepresentations. □

Notation 3.2.17. *Let us first fix some notation.*

- *When $\mathscr{D}$ is a subset of diagrams in NC^I for a certain set I, we denote by $\langle\mathscr{D}\rangle$ the set of all diagrams that can be obtained by usual tensor products, compositions and involutions of diagrams in $\mathscr{D}$ (see Definition 3.2.3).*
- *If Γ is finitely generated, we will only consider generating subsets S_Γ of Γ which are stable under inversion and the category NC^{S_Γ} (see Notation 3.2.9).*

- *We denote by $NC_{\infty}^{S_\Gamma} \subset NC^{S_\Gamma}$, the sub-tensor category where morphisms are the non-crossing diagrams satisfying the rules: for each block there is one element $g \in S_\Gamma$ such that the points of this block are decorated by elements $g^{\pm 1}$ and the sum of the exponents are equal on top and bottom.*

- *We denote by $NC_{\Gamma,S_\Gamma} \subset NC^{S_\Gamma}$, the sub-tensor category where the diagrams are decorated by elements $g_i, h_j \in S_\Gamma$ such that in each block $\prod_i g_i = \prod_j h_j$.*

- *We denote by $NC_\Gamma \subset NC^\Gamma$, the sub-tensor category where the diagrams are decorated by elements $g_i, h_j \in \Gamma$ such that in each block $\prod_i g_i = \prod_j h_j$.*

One can notice the following fact that we will use in the proof of the next theorem.

Proposition 3.2.18. *If Γ is a finitely generated (discrete) group $\Gamma = \langle S_\Gamma \rangle$, $|S_\Gamma| = p$, then the tensor categories $NC_{\infty}^{S_\Gamma}$ and $Tens\left((H_N^{\infty +})^{*p}, \{U^{(i)}\}_{i=1}^p\right)$ satisfy:*

$$Tens\left((H_N^{\infty +})^{*p}, \left\{U^{(i)}\right\}_{i=1}^p\right) = span\left\{Lin\langle NC_{\infty}^{S_\Gamma}\rangle\right\}.$$

Proof. This follows immediately from Proposition 3.2.15. □

We are now ready to prove the following theorem:

Theorem 3.2.19. *If Γ is a finitely generated (discrete) group $\Gamma = \langle S_\Gamma \rangle$,*

$$Tens(H_N^+(\Gamma), a(g) : g \in S_\Gamma) = span\{Lin\langle NC_{\Gamma,S_\Gamma}\rangle\}.$$

This theorem can be generalized in the following more concrete statement:

Theorem 3.2.20. *Let Γ be any (discrete) group. Then for all $g_1, \dots, g_k, h_1, \dots, h_l \in \Gamma$:*

$$\begin{aligned} &Hom_{H_N^+(\Gamma)}\left(a(g_1) \otimes \cdots \otimes a(g_k), a(h_1) \otimes \cdots \otimes a(h_l)\right) \\ &\qquad = span\left\{T_p : p \in NC_\Gamma(g_1, \dots, g_k; h_1, \dots, h_l)\right\}. \end{aligned}$$

where the sets $NC_\Gamma(g_1, \dots, g_k; h_1, \dots, h_l)$ are composed of non-crossing partitions having the property that, when putting the elements g_i and h_j on the upper and lower row respectively, then in each block we must have $\prod_{i=1}^k g_i = \prod_{j=1}^l h_j$.

Proof. First, let us notice that, in order to deduce Theorem 3.2.20 from Theorem 3.2.19, it is enough to consider the subgroup Γ' generated by the elements $g_i, h_j \in \Gamma$ and S_Γ

containing these elements $g_i, h_j \in \Gamma$. Indeed, $C(H_N^+(\Gamma'))$ is then a sub-Woronowicz-C^*-algebra of $C(H_N^+(\Gamma))$ and it suffices to determine the morphisms in the full sub-category of intertwiners in $H_N^+(\Gamma')$ to get Theorem 3.2.20.

We recall (see Proposition 3.1.2) that we have a morphism $\pi_1 : C(H_N^{\infty+})^{*p} \to C(H_N^+(\Gamma))$ given by

$$U_{i_1j_1}^{(r_1)} \dots U_{i_kj_k}^{(r_k)} \mapsto a_{i_1j_1}(g_{r_1}) \dots a_{i_kj_k}(g_{r_k}),$$

with $p = |S_\Gamma|$. We are going to determine the kernel of this morphism and apply Proposition 3.2.14 and Proposition 3.2.15 to describe the category

$$Tens\left(H_N^+(\Gamma), a(g) : g \in S_\Gamma\right).$$

We claim that the kernel of π_1 is generated by the relations:

- $U_{ij}^{(r_1)} \dots U_{ij}^{(r_k)} = U_{ij}^{(s_1)} \dots U_{ij}^{(s_l)}$ if $\prod_i g_{r_i} = \prod_i g_{s_i}$
- $U_{ij}^{(r)} U_{ik}^{(s)} = 0 = U_{ji}^{(r)} U_{ki}^{(s)}$ if $j \neq k$
- $\sum_i U_{ij}^{(r_1)} \dots U_{ij}^{(r_k)} = \sum_j U_{ij}^{(r_1)} \dots U_{ij}^{(r_k)} = 1$ if $\prod_i g_{r_i} = e$

Indeed, if one denotes I the associated ideal $I \subset C(H_N^{\infty+})^{*p}$, it is clear that $I \subset \ker(\pi_1)$. To prove the other inclusion, it is enough to prove that there is a morphism $s : C(H_N^+(\Gamma)) \to C(H_N^{\infty+})^{*p}/I$ such that $s \circ \pi_1 = q : C(H_N^{\infty+}) \to C(H_N^{\infty+})/I$ is the canonical quotient morphism. We define s as follows:

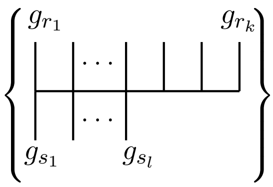
$$s(a_{ij}(g)) = q(U_{ij}^{(r_1)} \dots U_{ij}^{(r_k)})$$

for $g = \prod_i g_{r_i}$. The relations satisfied by the elements $a_{ij}(g)$ (see Example 3.1.1) are also clearly satisfied by the elements $q(U_{ij}^{(r_1)} \dots U_{ij}^{(r_k)})$ in $C(H_N^{+\infty})/I$. So by universality, s is well defined.

Moreover, s satisfies $s \circ \pi_1 = q$. Hence, $\ker(\pi_1) = I$ is generated by the relations presented above.

Before applying Proposition 3.2.14, we have to show that the relations generating $\ker(\pi_1)$ can be described by diagrams. With Notation (3.2.17), we claim that the one-block partitions $B_{k,l} \in NC_{\Gamma, S_\Gamma}, k, l \in \mathbb{N}$, decorated by elements certain $g_{r_1}, \dots, g_{r_k}$ and $g_{s_1}, \dots, g_{s_l}$ of S_Γ with $\prod_i g_{r_i} = \prod_i g_{s_i}$ describe these relations. Their pictorial representations are as follows

$$B_{k,l} = $$



$$\in NC_\Gamma(g_{r_1}, \dots, g_{r_k}; g_{s_1}, \dots, g_{s_l}).$$

The conditions

$$U_{ij}^{(r_1)} \dots U_{ij}^{(r_k)} = U_{ij}^{(s_1)} \dots U_{ij}^{(s_l)} \text{ with } \prod_i g_{r_i} = \prod_i g_{s_i}$$

follow from

$$T_{B_{k,l}} := T_{k,l} \in Hom\left(U^{(r_1)} \otimes \dots \otimes U^{(r_k)}; U^{(s_1)} \otimes \dots \otimes U^{(s_l)}\right).$$

More precisely, if (e_i) denotes the canonical basis of $\mathbb{C}^N$, we have

$$\begin{aligned}
&U^{(s_1)} \otimes \dots \otimes U^{(s_l)}(1 \otimes T_{k,l})(1 \otimes e_{p_1} \otimes \dots e_{p_k}) = (1 \otimes T_{k,l})U^{(r_1)} \otimes \dots \otimes U^{(r_k)}(1 \otimes e_{p_1} \otimes \dots e_{p_k}) \\
&\Leftrightarrow \delta_{p_1 = \dots = p_k} \sum_{i_1, \dots, i_l} U_{i_1 p_1}^{(s_1)} \dots U_{i_l p_1}^{(s_l)} \otimes e_{i_1} \otimes \dots \otimes e_{i_l} = (1 \otimes T_{k,l}) \sum_{i_1, \dots, i_k} U_{i_1 p_1}^{(r_1)} \dots U_{i_k p_k}^{(r_k)} \otimes e_{i_1} \otimes \dots \otimes e_{i_k} \\
&\Leftrightarrow \delta_{p_1 = \dots = p_k} \sum_{i_1, \dots, i_l} U_{i_1 p_1}^{(s_1)} \dots U_{i_l p_1}^{(s_l)} \otimes e_{i_1} \otimes \dots \otimes e_{i_l} = \sum_{i_1} U_{i_1 p_1}^{(r_1)} \dots U_{i_1 p_k}^{(r_k)} \otimes e_{i_1}^{\otimes l}.
\end{aligned}$$

i.e.

$$\begin{aligned}
&T_{k,l} \in Hom\left(U^{(r_1)} \otimes \dots \otimes U^{(r_k)}; U^{(s_1)} \otimes \dots \otimes U^{(s_l)}\right) \\
&\Leftrightarrow \begin{cases} U_{ij}^{(r_1)} \dots U_{ij}^{(r_k)} = U_{ij}^{(s_1)} \dots U_{ij}^{(s_l)} \\ \\ U_{ip_1}^{(r_1)} \dots U_{ip_k}^{(r_k)} = 0 \text{ if } p_t \neq p_s \text{ and } U_{i_1 p}^{(r_1)} \dots U_{i_k p}^{(r_k)} = 0 \text{ if } i_t \neq i_s, \text{ for some t,s.} \end{cases}
\end{aligned}$$

The last relations are equivalent to $U_{ij}U_{ik} = 0 = U_{ji}U_{ki}$ if $j \neq k$. Similar computations with $l = 0$ in the case $g_{r_1} \dots g_{r_k} = e$ give the relations

$$\sum_i U_{ij}^{(r_1)} \dots U_{ij}^{(r_k)} = \sum_j U_{ij}^{(r_1)} \dots U_{ij}^{(r_k)} = 1.$$

Notice that these one-block partitions $B_{k,l}$ generate with the usual tensor product, composition and involution operations all the diagrams in NC_{Γ, S_Γ}. Thus, with remarks following Notation 3.2.17, Proposition 3.2.14, Proposition 3.2.18 and the fact that $NC_\infty^{S_\Gamma} \subset NC_{\Gamma, S_\Gamma}$ we get:

$$Tens(H_N^+(\Gamma), a(g) : g \in S_\Gamma) = span\{Lin\langle NC_{\Gamma, S_\Gamma}, NC_\infty^{S_\Gamma}\rangle\} = span\{Lin\langle NC_{\Gamma, S_\Gamma}\rangle\}.$$

□

We get the following corollary concerning the basic corepresentations:

Corollary 3.2.21. *Let $N \geq 4$. The basic corepresentations of $C(H_N^+(\Gamma))$ satisfy:*

1. *The corepresentations $a(g), g \neq e$ are irreducible.*
2. *$a(e) = 1 \oplus \omega(e)$, with $\omega(e)$ an irreducible corepresentation.*
3. *The corepresentations $\omega(e), a(g), g \in \Gamma \setminus \{e\}$ are pairwise non-equivalent.*

Proof. We use the previous theorem and the fact that the linear maps T_p are linearly independent (so that we can identify these maps with the associated non-crossing partitions). Let $g, h \in \Gamma$, the previous theorem gives that

$$\dim(Hom(a(g), a(h)) = \#NC_\Gamma(g, h).$$

But it is easy to see that the only candidate elements in $NC_\Gamma(g, h)$ are

$$p = \left\{ \begin{matrix} g \\ | \\ h \end{matrix} \right\} \text{ and } q = \left\{ \begin{matrix} g \\ | \\ \\ | \\ h \end{matrix} \right\}$$

with the conditions $p \in NC_\Gamma(g, h) \Leftrightarrow g = h$ and $q \in NC_\Gamma(g, h) \Leftrightarrow g = h = e$. Now we can compute the cardinal $\#NC_\Gamma(g, h)$:

$$\#NC_\Gamma(g, h) = \begin{cases} 0 & \text{if } g \neq h \\ 1 & \text{if } g = h \neq e \\ 2 & \text{if } g = h = e \end{cases}$$

Then the second equality proves that the corepresentations $a(g), g \neq e$ are irreducible. The last equality, together with the fact that the trivial corepresentation is contained in $a(e)$ (since $\#NC_\Gamma(\emptyset; e) = 1$), prove that $a(e) = 1 \oplus \omega(e)$ where $\omega(e)$ is an irreducible corepresentation. And the fact that the basic corepresentations are pairwise non-equivalent comes from the first equality above. □

3.2.4 Fusion rules for $H_N^+(\Gamma)$

In this subsection, Γ denotes any (discrete) group, $N \geq 4$.

Definition 3.2.22. *The fusion semiring $(R^+, \bar{\ }, \oplus, \otimes)$ is defined as follows*

1. R^+ *is the set of equivalence classes of corepresentations.*

2. $(\ ^{-}\ , \oplus, \otimes)$ *are the usual involution, direct sum and tensor product of corepresentations.*

As in the Theorem 3.2.20, the irreducible representations of $H_N^+(\Gamma)$ can be indexed by the words over $\langle\Gamma\rangle$.

Definition 3.2.23. *Let $M = \langle\Gamma\rangle$ be the monoid formed by the words overs Γ. We endow M with the following operations:*

1. *Involution:* $(g_1, \dots, g_k)^- = (g_k^{-1}, \dots, g_1^{-1})$,

2. *concatenation: for any two words, we set*

$$(g_1, \dots, g_k), (h_1, \dots, h_l) = (g_1, \dots, g_{k-1}, g_k, h_1, h_2, \dots, h_l),$$

3. *Fusion: for two non-empty words, we set*

$$(g_1, \dots, g_k).(h_1, \dots, h_l) = (g_1, \dots, g_{k-1}, g_k h_1, h_2, \dots, h_l).$$

Notation 3.2.24. *If $(g_1, \dots, g_k) \in \langle\Gamma\rangle$, we will write $|(g_1, \dots, g_k)| = k$ to denote the length of the word $(g_1, \dots, g_k)$.*

Theorem 3.2.25. *The irreducible representations of $H_N^+(\Gamma)$ can be labelled $\omega(x)$ with $x \in M$, with involution $\overline{\omega}(x) = \omega(\overline{x})$ and the fusion rules:*

$$\omega(x) \otimes \omega(y) = \sum_{x=u,t\ ;\ y=\bar{t},v} \omega(u, v)\ \oplus \sum_{\substack{x=u,t\ ;\ y=\bar{t},v \\ u\neq\emptyset, v\neq\emptyset}} \omega(u.v)$$

and we have for all $g \in \Gamma$, $\omega(g) = a(g) \ominus \delta_{g,e} 1$.

We consider the set of irreducible corepresentations, the fusion semiring and the fusion ring of $H_N^+(\Gamma)$: $Irr(H_N^+(\Gamma)) \subset R^+ \subset R$. We also consider the additive monoid $\mathbb{N}\langle\Gamma\rangle$ with basis $B := \{b_x : x \in \langle\Gamma\rangle\}$, involution $\overline{b}_x = b_{\overline{x}}$ and fusion rules

$$b_x \otimes b_y = \sum_{x=u,t\ ;\ y=\bar{t},v} b_{u,v}\ + \sum_{\substack{x=u,t\ ;\ y=\bar{t},v \\ u\neq\emptyset, v\neq\emptyset}} b_{u.v}. \tag{3.8}$$

We want to prove that $\mathbb{N}\langle\Gamma\rangle \simeq R^+$ in such a way that B corresponds to $Irr(H_N^+(\Gamma))$. To do this, we are going to construct an isomorphism $\Phi : \mathbb{N}\langle\Gamma\rangle \to R^+$.

We construct and study Φ at the level of $\mathbb{Z}\langle\Gamma\rangle$ and R, where $\mathbb{Z}\langle\Gamma\rangle$ is the free $\mathbb{Z}$-module with basis $(b_x)_{x\in\langle\Gamma\rangle}$. Then $(\mathbb{Z}\langle\Gamma\rangle,+,\times)$ is a free ring over Γ for the product $b_x \times b_y = b_{x,y}$ where x, y is the concatenation of the words x, y.

Lemma 3.2.26. *$\mathbb{Z}\langle\Gamma\rangle$ is also a free ring for the product defined by the fusion rules above (3.8) and denoted $\otimes$.*

Proof. Indeed, consider the ring $\mathbb{Z}\langle X_g, g \in \Gamma\rangle$ of non-commutative polynomials with variables indexed by Γ, and

$$F : (\mathbb{Z}\langle X_g : g \in \Gamma\rangle, +, \times) \to (\mathbb{Z}\langle\Gamma\rangle, +, \otimes)$$

defined by $X_g \mapsto b_g$. This morphism is bijective:

For all $g_1, \dots, g_k \in \Gamma$, we have by (3.8):

$$b_{g_1,\dots,g_k} = b_{g_1,\dots,g_{k-1}} \otimes b_{g_k} \ominus b_{g_1,\dots,g_{k-1}g_k} \ominus \delta_{(g_{k-1}g_k,e)} b_{g_1,\dots,g_{k-2}} \tag{3.9}$$

Then an induction over the length of the words $g_1, \dots, g_k \in \langle\Gamma\rangle$ shows that F is surjective.

Now we prove the injectivity of the morphism F. Let $P \in \mathbb{Z}\langle X_g : g \in \Gamma\rangle$ with $F(P) = 0$. Suppose that $d := \deg(P) \geq 1$, i.e. that we can write

$$P = \sum \lambda(g_1, \dots, g_d) X_{g_1} \dots X_{g_d} + Q$$

with $\lambda(g_1, \dots, g_d) \neq 0$, $\deg(Q) < \deg(P)$. Then, we have :

$$\begin{aligned} 0 =& F(P) = \sum \lambda(g_1, \dots, g_d) b_{g_1} \otimes \cdots \otimes b_{g_d} + F(Q) \\ &= \sum \lambda(g_1, \dots, g_d) b_{g_1,\dots,g_d} + c \quad \left(c = \sum \mu_x b_x : |x| < d\right) \\ &\Rightarrow \lambda(x) = 0 \ \forall x \text{ with } |x| = d \text{ (since } B \text{ is a basis of } \mathbb{Z}\langle\Gamma\rangle). \end{aligned}$$

Thus we obtain $P = Q$ which contradicts $\deg(P) = d$. □

Then, $(\mathbb{Z}\langle\Gamma\rangle, +, \otimes)$ being a free ring, we can define a morphism $\Phi : \mathbb{Z}\langle\Gamma\rangle \to R$ by

$$\Phi(b_g) = \omega(g), g \in \Gamma.$$

We will prove that:

- $\Phi : \mathbb{Z}\langle\Gamma\rangle \to R$ is injective,
- $\Phi(B) \subset Irr(H_N^+(\Gamma))$,

- $\Phi : B \to Irr(H_N^+(\Gamma))$ is surjective.

We will both denote by $z \mapsto \#(1 \in z)$ the linear form counting:

- the number of copies of the trivial corepresentation contained in a $z \in R^+$,
- the coordinate of an element $z \in \mathbb{Z}\langle\Gamma\rangle$ relative to $b_\emptyset$.

We will denote by $b_x^*(w)$ the coordinate of $w \in \mathbb{Z}\langle\Gamma\rangle$ relative to b_x i.e. $\{b_x^* : x \in \langle\Gamma\rangle\}$ is the dual base of $\{b_x : x \in \mathbb{Z}\langle\Gamma\rangle\}$.

Lemma 3.2.27. *Φ commutes with the linear form $z \mapsto \#(1 \in z)$.*

Proof. By linearity, it is enough to check that we have

$$\#(1 \in b_{g_1} \otimes \cdots \otimes b_{g_k}) = \#(1 \in \omega(g_1) \otimes \cdots \otimes \omega(g_k)) \tag{3.10}$$

and it is equivalent to show that: $\forall g_1, \dots, g_k \in \Gamma$

$$\#\left(1 \in [(b_{g_1} + \delta_{g_1,e}1) \otimes \cdots \otimes (b_{g_k} + \delta_{g_k,e}1)]\right) = \#\left(1 \in a(g_1) \otimes \cdots \otimes a(g_k)\right)$$

by definitions of $\omega(g) = a(g) \ominus \delta_{g,e}1$.

Let us set $P_k := (b_{g_1} + \delta_{g_1,e}1) \otimes \cdots \otimes (b_{g_{k-1}} + \delta_{g_{k-1},e}1) \otimes (b_{g_k} + \delta_{g_k,e}1)$ and prove, by induction over k, the following statement,

$HR(k)$: "if $x = (x_1, \dots, x_l)$ is a certain sequence of products (ordered 1 to k) of elements of the set $\{g_1, \dots, g_k\}$, then

$$b_x^*(P_k) = \#NC'_\Gamma((x_1, \dots, x_l); (g_1, \dots, g_k)),$$

where $NC'_\Gamma((x_1, \dots, x_l); (g_1, \dots, g_k)) \subset NC_\Gamma((x_1, \dots, x_l); (g_1, \dots, g_k))$ is the sub-set composed of the elements $p \in NC_\Gamma((x_1, \dots, x_l); (g_1, \dots, g_k))$ with the additional rule that in each block there is at most one upper point and at least one lower point".

Since, by Theorem 3.2.20 and the fact that the linear maps T_p are linearly independent (see Theorem 3.2.12), we have

$$\#\left(1 \in a(g_1) \otimes \cdots \otimes a(g_k)\right) = \#NC_\Gamma(\emptyset; (g_1, \dots, g_k)),$$

then (3.10) will result from the case $x = \emptyset$ of $HR(k)$ (only lower points carrying elements g_i, with product e).

The case $k=1$ is easily proved. Indeed in this case, either $x=(g), g\in\Gamma$, or $x=\emptyset$. We have

$$b_g^*(b_g+\delta_{g,e}1)=1=\#NC'_\Gamma(g;g)$$

and

$$b_\emptyset^*(b_g+\delta_{g,e}1)=\left\{\begin{array}{ll}1 & \text{if } g=e\\ 0 & \text{otherwise}\end{array}\right\}=\#NC'_\Gamma(\emptyset;g).$$

Now assume that the result is proved for all elements of type $P_{k-1}=(b_{g_1}+\delta_{g_1,e}1)\otimes\cdots\otimes(b_{g_{k-1}}+\delta_{g_{k-1},e}1)$. Let $g_k\in\Gamma$ and $x=(x_1,\dots,x_l)$ be a sequence of products of elements in $\{g_1,\dots,g_k\}$. We consider $P_{k-1}\otimes(b_{g_k}+\delta_{g_k,e}1)$ and we first deal with the case $g_k\neq e$. We have

$$b_x^*(P_{k-1}\otimes b_{g_k})=\delta_{g_k,x_l}b^*_{(x_1,\dots,x_{l-1})}(P_{k-1})+b^*_{(x_1,\dots,x_lg_k^{-1})}(P_{k-1})+b^*_{(x_1,\dots,x_l,g_k^{-1})}(P_{k-1}).\tag{3.11}$$

The first term of (3.11) corresponds to the concatenation operation described by the fusion rules (3.8) and, by induction, it is equal to

$$\#NC'_\Gamma((x_1,\dots,x_{l-1});(g_1,\dots,g_{k-1})).$$

The concatenation of such non-crossing partitions with the one-block $p=\left\{\begin{array}{c}x_l\\ |\\ g_k\end{array}\right\}$ will give all the non-crossing partitions in $NC'_\Gamma((x_1,\dots,x_l);(g_1,\dots,g_k))$ where g_k is the only lower point in its block and connected to an upper point, $\{x_l\}=\{g_k\}$.

The second term of (3.11) corresponds to the fusion operation described in (3.8) and, by induction, it is equal to

$$\#NC'_\Gamma((x_1,\dots,x_{l-1},x_lg_k^{-1});(g_1,\dots,g_{k-1})).$$

These non-crossing partitions carry the upper point $\{x_lg_k^{-1}\}$ and thus, because of the definition of NC'_Γ, we have $x_l=(\prod_i g_i)g_k$ for some $g_i\in\{g_1,\dots,g_{k-1}\}$. We obtain this way, all the non-crossing partitions in $NC'_\Gamma((x_1,\dots,x_l);(g_1,\dots,g_k))$ where g_k is connected to some other lower points and to the upper point $\{x_l\}$.

The third and last term of (3.11), corresponds to the case where g_k is the inverse of the last term in the sequence $(x_i)_i$, by induction, it is equal to

$$\#NC'_\Gamma((x_1,\dots,x_l,g_k^{-1});(g_1,\dots,g_{k-1})).$$

These partitions carry the upper point g_k^{-1} and thus we have $(\prod_i g_i)g_k = e$ for some $g_i \in \{g_1, \dots, g_{k-1}\}$. We obtain, this way, all the non-crossing partitions in $NC'_\Gamma((x_1, \dots, x_l); (g_1, \dots, g_k))$ where g_k is connected to other lower points but to no upper point.

Altogether, we have proved

$$b_x^*(P_{k-1} \otimes b_{g_k}) = \#NC'_\Gamma((x_1, \dots, x_l); (g_1, \dots, g_k)).$$

In the case, $g_k = e$ we have

$$\begin{aligned} &b_x^*(P_{k-1} \otimes (b_e + 1)) \\ &= \left(\delta_{e,x_l} b^*_{(x_1,\dots,x_{l-1})}(P_{k-1}) + b^*_{(x_1,\dots,x_l)}(P_{k-1}) + b_{(x_1,\dots,x_l,e)}(P_{k-1})\right) + b^*_{(x_1,\dots,x_l)}(P_{k-1}), \end{aligned}$$

the additional term $b^*_{(x_1,\dots,x_l)}(P_{k-1})$ corresponding to the non-crossing partitions where $g_k = e$ is connected neither to another lower point, neither to an upper point. This case is not admissible if $g_k \neq e$. The rest of the proof is similar to the other case. □

We can now prove the theorem:

Proof of Theorem 3.2.25. We first prove that Φ is injective. Let $\alpha \in \mathbb{Z}\langle\Gamma\rangle$ in the domain of Φ. We denote by α^* the conjugate of α in $\mathbb{Z}\langle\Gamma\rangle$ (given on B by $\bar{b}_x = b_{\bar{x}}$). Then, we have by Lemma 3.2.27

$$\begin{aligned} \Phi(\alpha) = 0 &\implies \Phi(\alpha \otimes \alpha^*) = 0 \\ &\implies \#(1 \in \Phi(\alpha \otimes \alpha^*) = 0) \\ &\implies \#(1 \in \alpha \otimes \alpha^*) = 0 \text{ by } (3.10) \\ &\implies \alpha = 0. \end{aligned}$$

The last implication comes from the fact that $\alpha \to \#(1 \in \alpha \otimes \alpha^*)$ is a non-degenerate quadratic form, since for all words $w_1, w_2 \in \langle\Gamma\rangle$,

$$\#(1 \in b_{w_1} \otimes \overline{b_{w_2}}) = \#(1 \in b_{w_1} \otimes b_{\overline{w_2}}) = \delta_{\overline{w_2}, w_1}.$$

Now we prove that $\Phi(B) \subset Irr(H_N^+(\Gamma))$ by induction on the length of the words $x \in \langle\Gamma\rangle$. It is clear that for all letters $g \in \Gamma$, $\Phi(g) \in Irr(H_N^+(\Gamma))$ by Corollary 3.2.21.

Now for a word of length k, $x = g_{i_1}, \dots, g_{i_k}$. We have by (3.8):

$$b_{g_{i_1},\dots,g_{i_k}} = b_{g_{i_1}} \otimes b_{g_{i_2},\dots,g_{i_k}} - b_{g_{i_1}g_{i_2},g_{i_3},\dots,g_{i_k}} - \delta_{g_{i_1}g_{i_2},e} b_{g_{i_3},\dots,g_{i_k}} \in \mathbb{Z}\langle\Gamma\rangle, \tag{3.12}$$

and applying Φ, we get

$$\omega(g_{i_1},\dots,g_{i_k})=\omega(g_{i_1})\otimes\omega(g_{i_2},\dots,g_{i_k})-\omega(g_{i_1}g_{i_2},g_{i_3},\dots,g_{i_k})-\delta_{g_{i_1}g_{i_2},e}\omega(g_{i_3},\dots,g_{i_k})\in R.$$

We want to prove that it is an element of $Irr(H_N^+(\Gamma))$, we first prove that it is an element of R^+ and then that it is an irreducible corepresentation. To fulfill the first part, since Φ is injective, we only have to prove that

$$Hom\left(\omega(g_{i_1}g_{i_2},g_{i_3},\dots,g_{i_k});\omega(g_{i_1})\otimes\omega(g_{i_2},\dots,g_{i_k})\right)$$

and

$$Hom(\omega(g_{i_3},\dots,g_{i_k});\omega(g_{i_1})\otimes\omega(g_{i_2},\dots,g_{i_k}))$$

are one-dimensional. We have by the Frobenius reciprocity

$$\begin{aligned}
&dim\ Hom(\omega(g_{i_1}g_{i_2},g_{i_3},\dots,g_{i_k});\omega(g_{i_1})\otimes\omega(g_{i_2},\dots,g_{i_k}))\\
&=dim\ Hom(1,\overline{\omega(g_{i_1}g_{i_2},g_{i_3},\dots,g_{i_k})}\otimes\omega(g_{i_1})\otimes\omega(g_{i_2},\dots,g_{i_k}))\\
&=\#(1\in\overline{\omega(g_{i_1}g_{i_2},g_{i_3},\dots,g_{i_k})}\otimes\omega(g_{i_1})\otimes\omega(g_{i_2},\dots,g_{i_k}))\\
&=\#(1\in\overline{b_{g_{i_1}g_{i_2},g_{i_3},\dots,g_{i_k}}}\otimes b_{g_{i_1}}\otimes b_{g_{i_2},\dots,g_{i_k}})\\
&=\#\left(1\in b_{g_{i_k}^{-1},g_{i_{k-1}}^{-1},\dots,g_{i_2}^{-1}g_{i_1}^{-1}}\otimes b_{g_{i_1}}\otimes b_{g_{i_2},\dots,g_{i_k}}\right)=1.
\end{aligned}$$

The last equality comes from the facts that

- $1\in b_y\otimes b_x\Leftrightarrow y=\overline{x}$ (and in this case $\#(1\in b_y\otimes b_x)=1$),
- $b_{g_{i_1}}\otimes b_{g_{i_2},\dots,g_{i_k}}=\sum\lambda_x b_x$ with $|x|=k-1,\lambda_x\neq 0\Leftrightarrow x=(g_{i_1}g_{i_2},\dots,g_{i_k})$.

A similar computation shows that $Hom(\omega(g_{i_3},\dots,g_{i_k});\omega(g_{i_1})\otimes\omega(g_{i_2},\dots,g_{i_k}))$ is also one-dimensional in the case $g_{i_1}g_{i_2}=e$.

Finally, we can prove by similar arguments that

$$dim\ Hom(\omega(g_{i_1},\dots,g_{i_k});\omega(g_{i_1},\dots,g_{i_k}))=1$$

i.e. $\omega(g_{i_1},\dots,g_{i_k})$ is irreducible.

To conclude we must prove that $\Phi:B\to Irr(H_N^+(\Gamma))$ is surjective. An induction over k shows that for all $g_1,\dots,g_k\in\Gamma$, we have:

$$b_{g_1}\otimes\dots\otimes b_{g_k}=\sum_l\sum_{j_1,\dots,j_l}C_{j_1\dots j_l}b_{g_{j_1},\dots,g_{j_l}}$$

for some coefficients $C \in \mathbb{N}$ so that, applying Φ,

$$\omega(g_1) \otimes \cdots \otimes \omega(g_k) = \sum_l \sum_{j_1,\ldots,j_l} C_{j_1\ldots j_l} \Phi(b_{g_{j_1},\ldots,g_{j_k}}).$$

Then any tensor product between basic corepresentations $\omega(g), g \in \Gamma$ is in $\Phi(\mathbb{N}B) = span_{\mathbb{N}}\langle B\rangle$, and the surjectivity follows since the coefficients of such tensor products generate $C(H_N^+(\Gamma))$ so that $\Phi(B) \supset Irr(H_N^+(\Gamma))$. $\square$

We can give an alternative formulation of the description of these fusion rules: let a be the generator of the monoid $\mathbb{N}$ with respect to the operation $+$ and $(z_g)_{g\in\Gamma}$ be a family of abstract elements satisfying exactly all the relations of the group Γ. We put $M' = \mathbb{N} *_e \Gamma$, the free product identifying both neutral elements of $\mathbb{N}$ and Γ with the empty word. Then M' is the monoid generated by the element a and the family $(z_g)_{g\in\Gamma}$ with:

- involution: $a^* = a, z_g^* = z_{g^{-1}}$,
- (fusion) operation inductively defined by:

$$vaz_g \otimes z_h aw = vaz_{gh}aw + \delta_{gh,e}(v \otimes w), \tag{3.13}$$

- unit z_e.

Any element of M' can be written as a “reduced" word in the letters a, $z_g, g \in \Gamma$, $\alpha = a^{l_1} z_{g_1} a^{l_2} z_{g_2} \ldots a^{l_k}$ with

- $l_1, l_k \geq 0$ and $l_i \geq 1$ for all $1 < i < k$,
- $g_i \neq e$ for all i if $k > 1$,
- $\alpha = a^l$ in the case $k = 1$ for some $l \geq 0$ and a^0 is the empty word equal to z_e.

We obtain the following reformulation of the previous theorem:

Theorem 3.2.28. *The irreducible representations r_α of $H_N^+(\Gamma)$ can be indexed by the elements α of the submonoid $S := \langle az_g a : g \in \Gamma\rangle \subset M'$ and with fusion rules given by (3.13). Furthermore, the basic corepresentations $\omega(g), g \in \Gamma \setminus \{e\}$ correspond to the words $az_g a$, $\omega(e)$ to a^2 and the trivial one to $a^0 = 1$.*

Proof. We first use the identification proved in the previous theorem: $\omega(g_1, \ldots, g_k) \mapsto b_{g_1} \ldots b_{g_k}$. Then, for any words $x, y \in < \Gamma >$ and letters $g, h \in \Gamma$:

$$\begin{aligned}\omega(x,g) \otimes \omega(h,y) &= \sum_{\substack{x,g=u,t\\ h,y=\bar{t},v}} \omega(u,v) \oplus \omega(u.v)\\ &= \omega(x,g,h,y) \oplus \omega(x,gh,y) \oplus \delta_{gh,e}\omega(x) \otimes \omega(y)\end{aligned}$$

Then with the identification mentioned above, we obtain this new (recursive) formulation for the fusion rules:

$$pb_g \otimes b_h q = pb_g b_h q \oplus pb_{gh} q \oplus \delta_{gh,e} p \otimes q \tag{3.14}$$

(with the identifications $\omega(x) \equiv p, \omega(y) \equiv q$). Now, we consider the submonoid $S \subset M'$ generated by elements $az_g a : g \in \Gamma$. It is a free monoid indexed by Γ hence it is isomorphic to $\langle \Gamma \rangle$ (the monoid of the words over Γ introduced in the previous theorem) via $b_g \equiv az_g a$. We have $(az_g a)^* = az_{g^{-1}} a \in N$ so this identification is compatible with the involutions.

Now let $p, q \in S$. We prove that with the fusion rules (3.13) we can get back to (3.14), and then the identification will preserve the fusion rules. It comes as follows:

$$\begin{aligned}pb_g \otimes b_h q &\equiv paz_g a \otimes az_h aq\\ &= paz_g a^2 z_h aq \oplus paz_g \otimes z_h aq\\ &\equiv pb_g b_h q \oplus pb_{gh} q \oplus \delta_{gh,e} p \otimes q.\end{aligned}$$

□

3.2.5 Dimension formula

In this section we obtain the same dimension formula as in the case $\Gamma = \mathbb{Z}_s$ see [BV09, Theorem 9.3] and [Lem13b, Corollary 2.2]. In this subsection Γ is any (discrete) group, $N \geq 2$.

Let us first fix some notation. Recall that there is a morphism $\pi : C(H_N^+(\Gamma)) \to C(S_N^+)$ and that it corresponds a functor $\pi : Irr(H_N^+(\Gamma)) \to Irr(S_N^+)$, sending any $a(g), g \in \Gamma$ to the fundamental corepresentation v of S_N^+.

With the notation of Theorem 3.2.25 and Theorem 3.2.28 above, recall that if $r_\alpha \in Irr(H_N^+(\Gamma))$, we denote by $\chi_\alpha = (id \otimes Tr)(r_\alpha)$ the associated character.

It is proved in [Bra12b, Proposition 4.8] that the central algebra $C(S_N^+)_0 = C^* - \langle \chi_k : k \in \mathbb{N} \rangle$ is isomorphic with $C([0, N])$ via $\chi_k \mapsto A_{2k}(\sqrt{X})$ where $(A_k)_{k \in \mathbb{N}}$ is the family of dilated Tchebyshev polynomials defined inductively by $A_0 = 1, A_1 = X$ and $A_1 A_k = A_{k+1} + A_{k-1}$.

We now give the following proposition whose proof can be found in [Lem13b] since the fusion rules binding irreducible representations of $H_N^+(\Gamma)$ are similar for all groups Γ (and $N \geq 4$):

Proposition 3.2.29. *([Lem13b, Proposition 2.1]) Let χ_α be the character of an irreducible corepresentation $r_\alpha \in Irr(H_N^+(\Gamma))$. Write $\alpha = a^{l_1} z_{g_1} \dots a^{l_k}$. Then, identifying $C(S_N^+)_0$ with $C([0, N])$, the image of χ_α by π, say P_α, satisfies*

$$P_\alpha(X^2) = \pi(\chi_\alpha)(X^2) = \prod_{i=1}^{k} A_{l_i}(X).$$

Corollary 3.2.30. *([Lem13b, Corollary 2.2]) Let r_α be an irreducible representation of $H_N^+(\Gamma)$ with $\alpha = a^{l_1} z_{g_1} \dots a^{l_k}$. Then*

$$\dim(r_\alpha) = \prod_{i=1}^{k} A_{l_i}(\sqrt{N}).$$

3.3 Properties of the reduced operator algebra $C_r(H_N^+(\Gamma))$

3.3.1 Simplicity and uniqueness of the trace of $C_r(H_N^+(\Gamma))$

In this subsection, we will assume $N \geq 8$ and $|\Gamma| \geq 4$. The cases $|\Gamma| = 1, 2, 3$ correspond to S_N^+, H_N^+ and H_N^{3+}. The simplicity result proved in this subsection is already known (and we will use it) in the case of S_N^+ (see [Bra12b]). As for H_N^+, it is an easy quantum group and we do not investigate any further this subject in this thesis. We do not deal with the case $|\Gamma| = 3$, either.

The assumption on $N \geq 8$, is due to the fact that the simplicity of $C_r(S_N^+)$ is only known in the cases $N \geq 8$. For $N = 2$, we know (see [Bic04]) that $C_r(H_2^+(\Gamma)) \simeq C_r^*(\Gamma * \Gamma) \otimes C(\mathbb{Z}_2)$ which is not simple. To summarize, the cases $3 \leq N \leq 7$, $|\Gamma| = 2, 3$ remain open.

We first fix some notation. We use the description of the irreducible corepresentations indexed by the monoid M (see Theorem 3.2.25) but we will simplify the notation $\omega(g_1, \dots, g_k)$ into $(g_1, \dots, g_k)$ and will denote the empty word by 1 since it indexes the trivial representation. If $\alpha \in M$, we will denote the associated irreducible corepresentation by r_α. We will denote by $|\alpha| = |(g_1, \dots, g_k)|$ the length $k \in \mathbb{N}$ of $\alpha \in M$. If $A \subset M$, we denote by $\overline{A}$ the set of conjugates $\bar{\alpha}$ of the elements $\alpha \in A$.

We will use the following notation as in [Ban97]. If $A, B \subset M$, we set

$$A \circ B := \{\gamma : \exists(\alpha, \beta) \in A \times B \text{ such that } r_\gamma \subset r_\alpha \otimes r_\beta\} \subset M.$$

We denote by $(g, \dots) \in M$ an element starting by $g \in \Gamma$ and $(\dots, g)$ an element ending by g.

Notation 3.3.1. *We will denote e the neutral element in Γ and:*

- e^k *the word* $(e, \dots, e) \in M$, *with the convention* $e^0 = 1$.
- $E_1 := \bigcup\{(e, \dots)\} \cup \{1\}$ *the subset of the words starting by* e.
- $E_2 := \bigcup_{k\in\mathbb{N}}\{e^k\}$ *the subset of words with only* e *letters* ($E_2 \subset E_1$).
- $G_1 := \bigcup_{\substack{g\neq e\\ g\in\Gamma}}\{(g, \dots)\}$ *the subset of the words starting by any* $g \neq e$ *(notice that* $M = E_1 \sqcup G_1$*).*
- $G_2 := \bigcup_{g,g'\neq e}\{(g, \dots, g')\}$.
- $S := {}^cE_2$.
- $E_3 := S \cap E_1 = E_1 \setminus E_2$ *(notice that* $S = E_3 \sqcup G_1$*).*

The definition of E_2 and the fusion rules in Theorem 3.2.25, clearly show that $1 \in E_2$, $\overline{E_2} = E_2$ and $E_2 \otimes E_2 \subset E_2$. Let $\mathscr{C}'$ be the closure in $C_r(H_N^+(\Gamma))$ of the subspace $\mathscr{C}$ generated by the coefficients of the corepresentations $r_\alpha, \alpha \in E_2$. We know by [Ver04, Lemme 2.1, Proposition 2.2], that there exists a unique conditional expectation $P : C_r(H_N^+(\Gamma)) \twoheadrightarrow \mathscr{C}'$ such that the Haar state $h_{\mathscr{C}'}$ is the restriction of the Haar state $h \in C_r(H_N^+(\Gamma))^*$ and $h = h_{\mathscr{C}'} \circ P$.

We recall from [Ver04], that P is defined by the compression by the orthogonal projection p onto the closure of $\mathscr{C}'$ in $L^2(H_N^+(\Gamma))$: $P : C_r(H_N^+(\Gamma)) \to p\mathscr{C}'p \simeq \mathscr{C}'$.

We will denote by $\mathscr{S}'$ the closure of $\mathscr{S} := span\{x \in Pol(H_N^+(\Gamma)) : supp(x) \subset S\}$ in $C_r(H_N^+(\Gamma))$. We have the decomposition $C_r(H_N^+(\Gamma)) = \mathscr{C}' \oplus \mathscr{S}'$, $P|_{\mathscr{C}'} = id$ and $Ker(P) = \mathscr{S}'$.

We are going to prove that $\mathscr{C}'$ can be identified with $C_r(S_N^+)$, use the simplicity of $C_r(S_N^+)$ (when $N \geq 8$, see [Bra12b]) and adapt the "modified Powers method" in [Ban97] where the author proves the simplicity of $C_r(U_N^+)$.

Proposition 3.3.2. $\mathscr{C}' \simeq C_r(S_N^+)$.

Proof. We first notice that at the level of the full C^*-algebras, we have $C(S_N^+) \simeq \mathscr{C}'$ where the closure is taken in $C(H_N^+(\Gamma))$. Indeed, with the notation of the first section (see Definition 1.1.12, Example 3.1.1), we can construct by universal properties the following morphisms :

$$C(H_N^+(\Gamma)) \xrightarrow{\pi_1} C(S_N^+) \xrightarrow{\pi_2} C(H_N^+(\Gamma)),\ a_{ij}(g) \mapsto v_{ij} \mapsto a_{ij}(e).$$

Now notice that $\forall x \in C(S_N^+)$, $\pi_1 \circ \pi_2(x) = x$ and thus $C(S_N^+)$ is isomorphic with the (unital) sub-C^*-algebra of $C(H_N^+(\Gamma))$ generated by the elements $a_{ij}(e)$ i.e.

$$C(S_N^+) \simeq C^* - \langle x \in Pol(H_N^+(\Gamma)) : supp(x) \subset \{1, a(e)\}\rangle \subset C(H_N^+(\Gamma)).$$

But $\mathscr{A} := *_{\mathrm{alg}} - \langle x \in Pol(H_N^+(\Gamma)) : supp(x) \subset \{1, a(e)\}\rangle = \mathscr{C}$. Indeed : $a(e) = 1 \oplus r_{(e)} = 1 \oplus r_{e^1}$, thus the inclusion $\mathscr{A} \subset \mathscr{C}$ is clear. On the other hand, the coefficients of $1 = r_{e^0}$ and $r_{e^1} = a(e) \ominus 1$ are in $\mathscr{A}$. The inclusion $\mathscr{C} \subset \mathscr{A}$ then follows by induction since for all $k \in \mathbb{N}^*$, we have

$$e^k = e^{k-1} \otimes e^1 \ominus e^{k-1}.$$

Thus we obtain $\overline{\mathscr{C}}^{||\cdot||} \simeq C(S_N^+)$ and then an isomorphism at the level of the reduced C^*-algebras (see e.g. [Ver04]) $\mathscr{C}' \simeq C_r(S_N^+)$. $\square$

Notice that we also proved $\mathscr{C} \simeq Pol(S_N^+)$.

From now on all the closures are taken in the reduced C^*-algebra $C_r(H_N^+(\Gamma))$.

Let $J \lhd C_r(H_N^+(\Gamma))$ be an ideal and let us prove that J is either $\{0\}$ or $C_r(H_N^+(\Gamma))$. It is clear that $P(J)$ is an ideal in $\mathscr{C}'$. Hence, the simplicity of $\mathscr{C}' \simeq C_r(S_N^+)$ implies that $P(J) = \{0\}$ or $\mathscr{C}'$.

Let us first assume that $P(J) = \{0\}$. Then since $Ker(P) = \mathscr{S}'$ we have $J \subset \mathscr{S}'$.

But this is possible only if $J = \{0\}$. Indeed, since $1 \notin S$ we have $\mathscr{S}' \subset Ker(h)$. But now $x \in J$ then $x^*x \in J \subset \mathscr{S}' \subset Ker(h)$ i.e. $h(x^*x) = 0$. Hence, $x = 0$ since h is faithful on $C_r(H_N^+(\Gamma))$. We then have $P(J) = \{0\} \Rightarrow J = \{0\}$.

In the sequel, we assume that $P(J) = \mathscr{C}'$ and we prove that $J = C_r(H_N^+(\Gamma))$.

Since $P(J) = \mathscr{C}' \ni 1$, there exists $x \in J$ such that $x = 1 - z$ with $z \in \mathscr{S}'$. We write $z = z_0 + (z - z_0)$ with $z_0 \in \mathscr{S}$ and $||z - z_0||_r < 1/2$. Notice that we can assume that z and z_0 are hermitians even if this means that one takes the real part of z and approximates z_0 by hermitian elements.

We are going to prove that we can find a finite family $(b_i) \subset Pol(H_N^+(\Gamma))$ such that $C_r(H_N^+(\Gamma)) \ni w \mapsto \sum_i b_i w b_i^*$ is unital and completely positive and $||\sum_i b_i z_0 b_i^*||_r < 1/2$. We will then get

$$\begin{aligned}
||1 - \sum_i b_i x b_i^*||_r &= ||\sum_i b_i(1-x)b_i^*||_r = ||\sum_i b_i z b_i^*||_r \\
&\leq ||\sum_i b_i z_0 b_i^*||_r + ||\sum_i b_i(z-z_0)b_i^*||_r \\
&\leq ||\sum_i b_i z_0 b_i^*||_r + ||z-z_0||_r < 1,
\end{aligned}$$

since any unital and completely positive map is contractive. Thus $\sum_i b_i x b_i^* \in J$ will be invertible and then we will get $J = C_r(H_N^+(\Gamma))$.

Let $g_1, g_2, g_3 \in \Gamma \setminus \{e\}$ be three arbitrary chosen, pairwise different elements (recall that $|\Gamma| \geq 4$). With the Notation 3.3.1, we have the following proposition:

Proposition 3.3.3. *Let* $\alpha_1 := (g_1, e), \alpha_2 := (g_2, e), \alpha_3 := (g_3, e)$ *in* S. *Let* $G \subset S$ *finite. Then:*

1. $S = E_3 \sqcup G_1, G_2 \circ E_1 \cap E_1 = \emptyset, \{\alpha_t\} \circ G_1 \cap \{\alpha_s\} \circ G_1 = \emptyset, \forall t \neq s$,
2. $\bigcup_{t=1}^3 \{\alpha_t\} \circ G_2 \circ \{\overline{\alpha_t}\} \subset G_2$,
3. $\exists \alpha \in S$ *s.t.* $\{\alpha\} \circ G \circ \{\overline{\alpha}\} \subset G_2$.

Proof. The first assertion in (1) is clear. The second follows from the following computations for $g, g' \neq e$:

$$(g, \dots, g') \otimes (e, \dots) = (g, \dots, g', e, \dots) \oplus (g, \dots, g', \dots),$$

and the third from

$$(g_t, e) \otimes (g, \dots) = (g_t, e, g, \dots) \oplus (g_t, g, \dots)$$

for $g \neq e$ and $t = 1, 2, 3$.

The assertion (2) follows from the following computation for $g, g' \neq e$:

$$\begin{aligned}
(g_t, e) \otimes (g, \dots, g') \otimes (e, g_t^{-1}) &= ((g_t, e, g, \dots, g') \oplus (g_t, g, \dots, g')) \otimes (e, g_t^{-1}) \\
&= (g_t, e, g, \dots, g', e, g_t^{-1}) \oplus (g_t, e, g, \dots, g', g_t^{-1}) \oplus \\
&\oplus (g_t, g, \dots, g', e, g_t^{-1}) \oplus (g_t, g, \dots, g', g_t^{-1}).
\end{aligned}$$

For (3) consider $\alpha = (g, e, \dots, e)$ for any fixed $g = g_t$, $t = 1, 2, 3$, and where the number of letters e in the word α is greater than $m := \max\{|\beta| : \beta \in G\}$. For any $\gamma \in G$, we can write $\gamma = (e^{l-1}, h_l, \dots, h_k)$ with $h_l \neq e, 1 \leq l \leq k$. Then

$$\alpha \otimes \gamma = \bigoplus_{s=-(l-1)}^{l-1} (g, e^{m+s}, h_l, \dots, h_k) \text{ since } l-1 < k \leq m$$

Now write $\gamma = (e^{l-1}, h_l, \dots, h_{l'}, e^{k-l'})$ with $l \leq l' \leq k$ and compute

$$\begin{aligned}\alpha \otimes \gamma \otimes \bar{\alpha} &= (g, e_m) \otimes \gamma \otimes (e_m, g^{-1}) \\ &= \bigoplus_{r=-(k-l')}^{k-l'} \bigoplus_{s=-(l-1)}^{l-1} (g, e^{m+s}, h_l, \dots, h_{l'}, e^{r+m}, g^{-1})\end{aligned}$$

Hence, $\{(g, e^m)\} \circ G \circ \{(e^m, g^{-1})\} \subset G_2$. $\square$

We shall apply Proposition 3.3.3 to $G = supp(z_0)$ (with $z_0 \in \mathscr{S}$ as above Proposition 3.3.3). We will then get that $z' = \sum_i a_i z_0 a_i^*$ has its support in G_2, where $(a_i) \subset Pol(H_N^+(\Gamma))$ is a finite family of coefficients of r_α with $\alpha \in S$ obtained in the assertion (3) of the previous proposition. The proof of the simplicity of $C_r(H_N^+(\Gamma))$ will then rely on a result proved in [Ban97] that we recall below.

We recall in Section 3.1 the construction of the adjoint representation of a compact quantum group of Kac type. We obtained for all irreducible characters χ_r, a completely positive map $ad(\chi_r) \in B(C_r(H_N^+(\Gamma)))$, such that

$$ad(\chi_r)(z) = \sum_{i,j} r_{ij} z r_{ij}^*.$$

We will simply denote $ad(r) := ad(\chi_r)$, if $r \in Irr(H_N^+(\Gamma))$. We then easily see that $ad(r)(1) = dim(r)1$ and $\tau(ad(r)(z)) = dim(r)\tau(z)$ for any trace $\tau \in C_r(H_N^+(\Gamma))^*$. Notice that with these notation, we have $z' = ad(r_\alpha)(z_0)$.

We put $d_t = dim(r_{\alpha_t})$ with $\alpha_t, t = 1, 2, 3$ defined in Proposition 3.3.3. Then, if one considers the maps $\dfrac{ad(r_{\alpha_t})}{d_t}$, one can get as in [Ban97, Proposition 8]:

Proposition 3.3.4. *The unital and completely positive linear map* $T : C_r(H_N^+(\Gamma)) \to C_r(H_N^+(\Gamma))$, $T = \sum_{t=1}^3 \dfrac{ad(\alpha_t)}{3d_t}$ *is such that:*

1. $T(z) = \sum_i a_i z a_i^*$ *for some finite family* $(a_i) \subset Pol(H_N^+(\Gamma))$.
2. T *is* τ*-preserving for any trace* $\tau \in C_r(H_N^+(\Gamma))^*$ *(hence* h*-preserving).*

3. *For all* $z = z^* \in C_r(H_N^+(\Gamma))$ *with* $supp(z) \circ E_1 \cap E_1 = \emptyset$*, we have* $||T(z)||_r \leq 0.95||z||_r$ *and* $supp(T(z)) \subset \bigcup_t \{\alpha_t\} \circ supp(z) \circ \{\overline{\alpha_t}\}$*.*

Then, we can get the simplicity of $C_r(H_N^+(\Gamma))$.

Theorem 3.3.5. *$C_r(H_N^+(\Gamma))$ is simple with unique trace h, for all $N \geq 8$ and any discrete group Γ, $|\Gamma| \geq 4$.*

Proof. We denote by τ any faithful normal trace on $C_r(H_N^+(\Gamma))$.

We recall that we assume now $P(J) = \mathscr{C}'$ (we already proved above that $P(J) = \{0\} \Rightarrow J = \{0\}$). By the discussion before Proposition 3.3.3, it remains to prove that there exists a finite family $(b_i) \subset Pol(H_N^+(\Gamma))$ such that $||\sum_i b_i z_0 b_i^*||_r < 1/2$.

We first apply Proposition 3.3.4 to the hermitian $z' = ad(r_\alpha)(z_0)$ with α obtained in Proposition 3.3.3. Notice that $ad(r_\alpha)(z_0) = \sum_i a_i z_0 a_i^* \in \mathscr{S}$. We then get a unital τ-preserving linear map $V_1 : C_r(H_N^+(\Gamma)) \to C_r(H_N^+(\Gamma))$ of the form $z \mapsto \sum_i c_i z c_i^*$ ($(c_i) \subset Pol(H_N^+(\Gamma))$ finite) with

$$||V_1(z')||_r \leq 0.95||z'||_r$$

since $supp(z') \subset G_2$ and $G_2 \circ E_1 \cap E_1 = \emptyset$. Moreover,

$$supp(V_1(z')) \subset \cup_t \{\alpha_t\} \circ supp(z') \circ \{\overline{\alpha_t}\} \subset G_2.$$

Then the τ-preserving map $z \mapsto V_1 ad(r_\alpha)(z)$ is of the form $z \mapsto \sum_i d_i z d_i^*$ for some (finite family of elements) $d_i \in Pol(H_N^+(\Gamma))$, and satisfies

$$V_1(ad(r_\alpha)(z_0))^* = V_1(ad(r_\alpha)(z_0)).$$

Thus we can apply the same arguments we just used as many times as needed so that there exists m such that

$$||V_m \dots V_1 ad(r_\alpha)(z_0)||_r < 1/2.$$

We put $V := V_m \dots V_1 ad(r_\alpha)$, which is a completely positive, unital map in $B(C_r(H_N^+(\Gamma)))$ of the form

$$z \mapsto \sum_i b_i z b_i^*,$$

where $(b_i) \subset Pol(H_N^+(\Gamma))$ is finite and $||\sum_i b_i z_0 b_i^*||_r < 1/2$.

The simplicity of $C_r(H_N^+(\Gamma))$ then follows from these observations and the discussion before Proposition 3.3.3.

For the uniqueness of the trace let us first show that if τ is any faithful normal trace on $C_r(H_N^+(\Gamma))$ then the restriction to $\mathscr{S}'$ is the restriction of the Haar state h. Indeed, let $x = x^* \in \mathscr{S}$. Then, on the one hand we have $h(x) = 0$ and on the other hand if $\epsilon > 0$, we can apply the method above to find a finite family $(a_i) \subset Pol(H_N^+(\Gamma))$ such that $y = \sum_i a_i x a_i^*$ has norm less than $\epsilon > 0$ and thus $|\tau(x)| = |\tau(y)| \leq \epsilon$. Letting $\epsilon \to 0$, we get that h and τ coincide on the hermitians of $\mathscr{S}$. But any $x \in \mathscr{S}$, is the sum of two hermitians, we then get $\tau|_{\mathscr{S}} = h|_{\mathscr{S}}$. By continuity, we get $\tau|_{\mathscr{S}'} = h|_{\mathscr{S}'}$.

Now take any $x \in C_r(H_N^+(\Gamma))$ and write $x = y + z$ with $y \in \mathscr{C}'$ and $z \in \mathscr{S}'$. One has

$$\tau(y+z) = \tau(y) + h(z) = h_{S_N^+}(y)$$

by the uniqueness of the trace on $C_r(S_N^+)$ and what we just proved on $\mathscr{S}'$. We then get $\tau(x) = h_{S_N^+}(y) = h(x)$ hence $\tau = h$.

$\square$

In particular there is a unique faithful normal trace on the von Neumann algebra $L^\infty(H_N^+(\Gamma))$, given by the extension of the Haar state h i.e.:

Corollary 3.3.6. *$L^\infty(H_N^+(\Gamma))$ is a II_1-factor for all $N \geq 8$, and any discrete group $|\Gamma| \geq 4$.*

3.3.2 Fullness of the II_1-factor $L^\infty(H_N^+(\Gamma))$

In this subsection, N is again an integer greater than 8 and Γ is any discrete group $|\Gamma| \geq 2$. Once more the case $3 \leq N \leq 7$ remain open. If $N = 2$, we have that $L^\infty(H_2^+(\Gamma)) = L(\Gamma * \Gamma \times \mathbb{Z}_2)$ and it is not a factor since $\Gamma * \Gamma \times \mathbb{Z}_2$, containing $\mathbb{Z}_2$ in its center, is not icc . We will denote by $||.||_2$ the $L^2(H_N^+(\Gamma))$-norm with respect to the tracial Haar state h and by $||.||_r$ the norm on $C_r(H_N^+(\Gamma))$.

Definition 3.3.7. *Let (M, τ) be a II_1-factor with unique faithful normal trace τ. A sequence $(x_n) \subset M$ is said to be asymptotically central if for all $y \in M$, $||x_n y - y x_n||_2 \to 0$. We say that (x_n) is asymptotically trivial if $||x_n - \tau(x_n)1||_2 \to 0$. The II_1-factor (M, τ) is said to be full if every bounded asymptotically central sequence is trivial.*

We use the decomposition of the preceding subsection 3.3.1, $Pol(H_N^+(\Gamma)) = \mathscr{C} \oplus \mathscr{S}$ which gives $L^\infty(H_N^+(\Gamma)) = \overline{\mathscr{C}}^{\sigma-w} \oplus \overline{\mathscr{S}}^{\sigma-w}$.

We want to prove the fullness of $L^\infty(H_N^+(\Gamma))$. It is easy to see that it is enough to consider sequences in the dense subalgebra $Pol(H_N^+(\Gamma))$. We then fix a bounded asymptotically central sequence $(x_n) \subset Pol(H_N^+(\Gamma)$. One can write $x_n = y_n + z_n$, with $y_n \in \mathscr{C}, z_n \in \mathscr{S}$. If $a \in \overline{\mathscr{C}}^{\sigma-w}$, we have

$$\begin{aligned} ||y_n a - a y_n||_{2,L^2(\mathscr{C}')} &= ||y_n a - a y_n||_2 \\ &= ||P(x_n)a - aP(x_n)||_2 \\ &= ||P(x_n a) - P(a x_n)||_2 \leq ||x_n a - a x_n||_2 \end{aligned}$$

where the last 2-norms are the $L^2(H_N^+(\Gamma))$-norm. The first equality above comes from the fact that the restriction to $\mathscr{C}'$ of the Haar state h of $C_r(H_N^+(\Gamma))$ is the Haar state on $\mathscr{C}'$. We then get $||y_n a - a y_n||_2 \to 0$, $\forall a \in \overline{\mathscr{C}}^{\sigma-w} \simeq L^\infty(S_N^+)$. As a result, we obtain that $||y_n - h(y_n)1||_2 \to 0$ since $L^\infty(S_N^+)$ is a full factor for $N \geq 8$ (see [Bra12b]). In particular, (y_n) is asymptotically central in $L^\infty(H_N^+(\Gamma))$, and hence this is also the case of (z_n).

Of course, this implies that $(z_n) \subset \mathscr{S}$ is also a central sequence in $L^\infty(H_N^+(\Gamma))$.

To get the fulness of $L^\infty(H_N^+(\Gamma))$ it remains to prove that (z_n) is asymptotically trivial. To do this we adapt the "$14-\epsilon$ method" introduced, in particular, to prove that $L(F_n)$ has not the property Γ and used also by Vaes to prove the fulness of $L^\infty(U_N^+)$ (see [DCFY13]).

Once more, we use the decomposition $S = E_3 \sqcup G_1$ which gives two orthogonal subspaces in $L^2(H_N^+(\Gamma))$:

$$H_1 = \overline{span}^{||\cdot||_2}\{\Lambda_h(x) : supp(x) \subset E_3\},$$

$$H_2 = \overline{span}^{||\cdot||_2}\{\Lambda_h(x) : supp(x) \subset G_1\}$$

where Λ_h is the GNS map associated to the Haar state h. Let us set $H := H_1 \perp H_2$, and H_1^0, H_2^0 the corresponding subspaces before taking closures. If $x \in L^\infty(H_N^+(\Gamma))$, we will simply write $x \equiv \Lambda_h(x)$ via $L^\infty(H_N^+(\Gamma)) \hookrightarrow L^2(H_N^+(\Gamma))$.

If $\beta \in S$, we set $d_\beta := dim(r_\beta)$ and $K_\beta := L^2(B(H_\beta), \frac{1}{d_\beta}Tr(\cdot))$, where H_β is the representation space of $r_\beta \in \mathscr{S} \otimes B(H_\beta)$, and we consider the isometry:

$$v_\beta : H \to H \otimes K_\beta, v_\beta a = r_\beta(a \otimes 1)r_\beta^* = r_\beta(a \otimes 1)r_{\overline{\beta}}.$$

We will denote the norm and scalar product on $L^2(H_N^+(\Gamma))$ by $||.||_2, \langle\cdot,\cdot\rangle_2$ and the scalar product and norm on the tensor spaces $H \otimes K_\beta$ simply by $||.||, \langle\cdot,\cdot\rangle$.

We fix an element $g \in \Gamma$, $g \neq e$. Remark that $g \neq e$ implies that we must assume $|\Gamma| \geq 2$. The case $|\Gamma| = 1$ corresponds to S_N^+ and the result we want to prove is already known in this case.

We recall that we denote by e^k, the word $(e,\dots,e)$ with k letters equal to e.

Lemma 3.3.8. *With the notation above, we have*

1. $\forall\beta\in E_3$, $\{(g)\}\circ\beta\circ\left\{\overline{(g)}\right\}\subset G_1$,
2. $\{e^i\}\circ G_1\circ\{e^i\}\subset E_3, i=2,4$,
3. $\{e^2\}\circ G_1\circ\{e^2\}\cap\{e^4\}\circ G_1\circ\{e^4\}=\emptyset$.

Proof. Let $\beta\in E_3$, i.e. $\beta=(e,h_1,\dots,h_l)$ with $l\geq 1$ and $h_i\neq e$ at least for one $i\in\{1,\dots,l\}$, we have:

$$\begin{aligned}(g)\otimes\beta\otimes\overline{(g)}&=(g,e,h_1,\dots,h_l)\otimes\overline{(g)}\oplus(g,h_1,\dots,h_l)\otimes\overline{(g)}\\&=(g,e,h_1,\dots,h_l,g^{-1})\oplus(g,e,h_1,\dots,h_lg^{-1})+\delta_{h_lg^{-1},e}(g,e,h_1,\dots,h_{l-1})\\&\oplus(g,h_1,\dots,h_l,g^{-1})\oplus(g,h_1,\dots,h_lg^{-1})+\delta_{h_lg^{-1},e}(g,h_1,\dots,h_{l-1})\end{aligned}$$

and then (1) follows.

Now let $\alpha\in G_1$, i.e. $\alpha=(h_1,\dots,h_l)$ with $h_1\neq e$. We have

$$\begin{aligned}(e^i)\otimes\alpha\otimes(e^i)&=((e^i,h_1,\dots,h_l)\oplus(e^{i-1},h_1,\dots,h_l))\otimes(e^i)\\&=(e^i,h_1,\dots,h_l,e^i)\oplus(e^i,h_1,\dots,h_l,e^{i-1})+\delta_{h_l,e}(e^i,h_1,\dots,h_{l-1})\otimes(e^{i-2})\\&\oplus(e^{i-1},h_1,\dots,h_l,e^i)\oplus(e^{i-1},h_1,\dots,h_l,e^{i-1})+\delta_{h_l,e}(e^{i-1},h_1,\dots,h_{l-1})\otimes(e^{i-2})\end{aligned}$$

and this gives $(2),(3)$. $\square$

Proposition 3.3.9. *For all $z\in\mathscr{S}$, we have*

$$||z||_2\leq 14\max\{||z\otimes 1-v_{(g)}z||,||z\otimes 1-v_{e^2}z||,||z\otimes 1-v_{e^4}z||\}.$$

Proof. We write $z=x+y$ with $x\in H_1^0$ and $y\in H_2^0$. We put $z':=v_{(g)}z, x':=v_{(g)}x$. Notice that by the relation (1) of Lemma 3.3.8 we have $v_{(g)}x\in H_2\otimes K_{(g)}$. In particular $\langle x',x\otimes 1\rangle=0$ if $\langle\cdot,\cdot\rangle$ denotes the scalar product on $H\otimes K_{(g)}$. We have:

$$\langle z\otimes 1-x\otimes 1-x',x'\rangle=\langle z\otimes 1-x',x'\rangle=\langle z\otimes 1-z',x'\rangle \tag{3.15}$$

where the last equality comes from the fact that $\langle v^1_{(g)}y,x'\rangle=\langle y,x\rangle_2=0$ since $v_{(g)}$ is an isometry. Thus we have

$$|\langle z\otimes 1-x\otimes 1-x',x'\rangle|\leq||z\otimes 1-z'||\;||x'||=||z\otimes 1-z'||\;||x||_2.$$

We get,

$$\begin{aligned}||x||_2^2+||y||_2^2=||z||_2^2&=||z\otimes 1-x\otimes 1-x'+x\otimes 1+x'||^2\\&=||z\otimes 1-x\otimes 1-x'||^2+||x||_2^2+||x'||^2-2Re\langle z\otimes 1-x\otimes 1-x',x'\rangle\\&\geq ||z\otimes 1-x\otimes 1-x'||^2+||x||_2^2+||x'||^2-2||z\otimes 1-z'||\ ||x||_2\end{aligned}$$

by (3.15) and $\langle z\otimes 1-x\otimes 1-x', x\otimes 1\rangle=0$ (since $\langle x', x\otimes 1\rangle=0$ and $x\in H_1, y\in H_2$). Then we obtain

$$||y||^2\geq ||x||_2^2-2||x||_2||z\otimes 1-v_{(g)}z||. \tag{3.16}$$

Now, we consider the isometries

$$v^2: H\to H\otimes K_{e^2}\otimes K_{e^4} \text{ defined by } v^2\xi=(v_{e^2}\xi)_{12},$$

$$v^4: H\to H\otimes K_{e^2}\otimes K_{e^4} \text{ defined by } v^4\xi=(v_{e^4}\xi)_{13},$$

We have by (2), (3) of Lemma 3.3.8

$$v^iH_2\subset H_1\otimes K_{e^2}\otimes K_{e^4},\ i=2,4,$$

$$v^2H_2\perp v^4H_2$$

We set

$$Y_2=v^2y, Y_3=v^4y,$$

$$Z_2=v^2z, Z_3=v^4z.$$

Then, $Y_2, Y_3\in H_1\otimes K_{e^2}\otimes K_{e^4}$ and Y_2, Y_3 are perpendicular. Notice that this implies that the vectors $y\otimes 1\otimes 1$, Y_2 and Y_3 are pairwise perpendicular in $H\otimes K_{e^2}\otimes K_{e^4}$ since $y\in H_2$.

Consider $X:=z\otimes 1\otimes 1-y\otimes 1\otimes 1-Y_2-Y_3$ and notice that X is perpendicular to $y\otimes 1\otimes 1$. Now we compute the scalar product:

$$\langle X, Y_2\rangle=\langle z\otimes 1\otimes 1-Y_2, Y_2\rangle=\langle z\otimes 1\otimes 1-Z_2, Y_2\rangle,$$

since $\langle v^2x, Y_2\rangle_2=\langle x,y\rangle_2=0$. Hence

$$|\langle X, Y_2\rangle|\leq ||y||_2||z\otimes 1\otimes 1-Z_2||.$$

Similarly, we have

$$|\langle X, Y_3\rangle|\leq ||y||_2||z\otimes 1\otimes 1-Z_3||.$$

We obtain, since $\langle X, y \otimes 1 \otimes 1\rangle = 0$,

$$\begin{aligned}||x||_2^2 + ||y||_2^2 = ||z||_2^2 &= ||X + y \otimes 1 \otimes 1 + Y_2 + Y_3||^2 \\ &\geq ||X||^2 + 3||y||_2^2 - 2||y||_2||z \otimes 1 \otimes 1 - Z_2||^2 - 2||y||_2||z \otimes 1 \otimes 1 - Z_3||^2\end{aligned}$$

and one can deduce that

$$||x||_2^2 \geq 2||y||_2^2 - 2||y||_2||z \otimes 1 - v_{e^2}z|| - 2||y||_2||z \otimes 1 - v_{e^4}z||. \tag{3.17}$$

We denote by $A = ||z \otimes 1 - v_{(g)}z||, B = ||z \otimes 1 - v_{e^2}||$ and $C = ||z \otimes 1 - v_{e^4}||$ and we get, applying (3.16), the fact that $||x||_2, ||y||_2 \leq ||z||_2$ and (3.17),

$$\begin{aligned}||y||_2^2 \geq ||x||_2^2 - 2||x||_2 A &\geq ||x||_2^2 - 2||z||_2 A \\ &\geq 2||y||_2^2 - 2||z||_2(A + B + C).\end{aligned}$$

We then obtain $||y||_2^2 \leq 2||z||_2(A + B + C)$.

On the other hand, applying (3.17), the fact that $||x||_2, ||y||_2 \leq ||z||_2$ and (3.16), we get

$$||x||_2^2 \geq 2||y||_2^2 - 2||z||_2(B + C) \geq 2||x||_2^2 - 2||z||_2(2A + B + C).$$

We then obtain $||x||_2 \leq 2||z||_2(2A + B + C)$ and we can conclude:

$$||z||_2^2 \leq 14||z||_2 \max\{A, B, C\}.$$

□

Theorem 3.3.10. *The II_1-factor $L^\infty(H_N^+(\Gamma))$ is full for all discrete groups Γ and all $N \geq 8$.*

Proof. This follows immediately from the previous proposition and the discussion before it. □

Remark 3.3.11. *The fact that any bounded asymptotically central sequence is asymptotically trivial implies that the center of $L^\infty(H_N^+(\Gamma))$ is trivial, and thus we get that $L^\infty(H_N^+(\Gamma))$ is a factor for all $N \geq 8$ and all discrete groups Γ.*

3.3.3 Haagerup approximation property for the dual of $H_N^+(\Gamma) = \widehat{\Gamma} \wr_* S_N^+$, Γ finite

In this subsection, Γ is a finite group and $N \geq 4$. We set $\mathbb{G} = H_N^+(\Gamma)$. If $\alpha \in Irr(\mathbb{G})$, let $L^2_\alpha(\mathbb{G}) := span\{\Lambda_h(x) : supp(x) = \alpha\} \subset L^2(\mathbb{G})$ where Λ_h is the GNS map associated to the Haar state h of $\mathbb{G}$.

Using the fundamental result in [Bra12b, Theorem 3.7], we can produce a net of normal, unital, completely positive h-preserving maps on $L^\infty(\mathbb{G})$ given by

$$T_{\phi_x} = \sum_{\alpha \in Irr(\mathbb{G})} \frac{\phi_x(\chi_{\overline{\alpha}})}{d_\alpha} P_\alpha.$$

In this formula, $\phi_x = ev_x \circ \pi$ with ev_x the evaluation map in $x \in I_N = [4, N], N \geq 5$ ($I_4 = [0, 4]$) on functions of $C(S_N^+)_0 \simeq C([0, N])$, π is the canonical map

$$\pi : C(H_N^+(\Gamma)) \to C(S_N^+)$$

and $P_\alpha : L^2(\mathbb{G}) \to L^2(\mathbb{G})_\alpha$ is the orthogonal projections associated to $\alpha \in Irr(\mathbb{G})$.

We introduce a proper function on the monoid S (see Theorem 3.2.28). Let L be defined by $L(\alpha) = \sum_{i=1}^{k_\alpha} l_i$ for $\alpha = a^{l_1} z_{g_1} \dots a^{l_{k_\alpha}}$ with $g_1, \dots, g_{k-1} \neq e$. Notice that, if Γ is finite, for all $R > 0$ the set $B_R = \left\{\alpha = a^{l_1} z_{g_1} \dots a^{l_{k_\alpha}} : L(\alpha) = \sum_{i=1}^{k_\alpha} l_i \leq R\right\} \subset S$ is finite. Thus we get that

$$\text{a net } (f_\alpha)_{\alpha \in S} \in c_0(S) \Longleftrightarrow \forall \epsilon > 0\ \exists R > 0 : \forall \alpha \in S, (L(\alpha) > R \Rightarrow |f_\alpha| < \epsilon).$$

We say that a net $(f_\alpha)_\alpha$ converges to 0 as $\alpha \to \infty$ if $(f_\alpha)_\alpha \in c_0(S)$. One can prove, as in [Lem13b, Proposition 3.3, Proposition 3.4], that the net $\left(\frac{\phi_x(\chi_{\overline{\alpha}})}{d_\alpha}\right)_{x \in I_N}$ converges to 0 as $\alpha \to \infty$ so that the extensions $T_{\phi_x} : L^2(H_N^+(\Gamma)) \to L^2(H_N^+(\Gamma))$ are compact operators. The pointwise convergence to the identity of these operators, in 2-norm, can be proved as in [Lem13b, Theorem 3.5]. Then:

Theorem 3.3.12. *The dual of $H_N^+(\Gamma) = \widehat{\Gamma} \wr_* S_N^+$ has the Haagerup property for all finite groups Γ and $N \geq 4$.*

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